## Mechanics.

The branch of applied mathematics which deals with the action of forces on bodies which we encounter in nature and technology. It is divided into statics and dynamics.

### Statics.

Statics deals with the bodies at rest.

### Dynamics.

Dynamics deals with bodies in motion. It can be further subdivided into kinematics and kinetics.

### Kinematics.

Kinematics deals with the motion of bodies without taking into account their masses or the forces acting on them.

### Kinetics.

Kinetics deals with the relations between the forces acting on bodies and the resulting motion.

### Classical Mechanics.

The major field of mechanics which deals with the particle moving with the speed less than the speed of light is called classical mechanics.

It deals with macroscopic objects.

## Quantum Mechanics.

The motion of objects sufficiently sma
is studied in quantum mechanics that
discuss the atomic nature of matter
with velocities comparable with the velo
of light.

## Relativistic Mechanics

Relativistic mechanics concerned wit
the motion of bodies whose relative
velocities approach the speed of light c

## Relative Velocity

Let A and B be two objects moving
uniform velocities $\vec{V_1}$ and $\vec{V_2}$. Then rela
velocity of object A w.r.t. B is $\vec{V_1}-$

## Relativistic Velocity

Any velocity that is sufficiently high to
signified changes in mass (or length or t
of the object is called relativistic veloc

## Inertial frame of reference

The frame of reference in which Newton's
are valid is called inertial frame of reference.

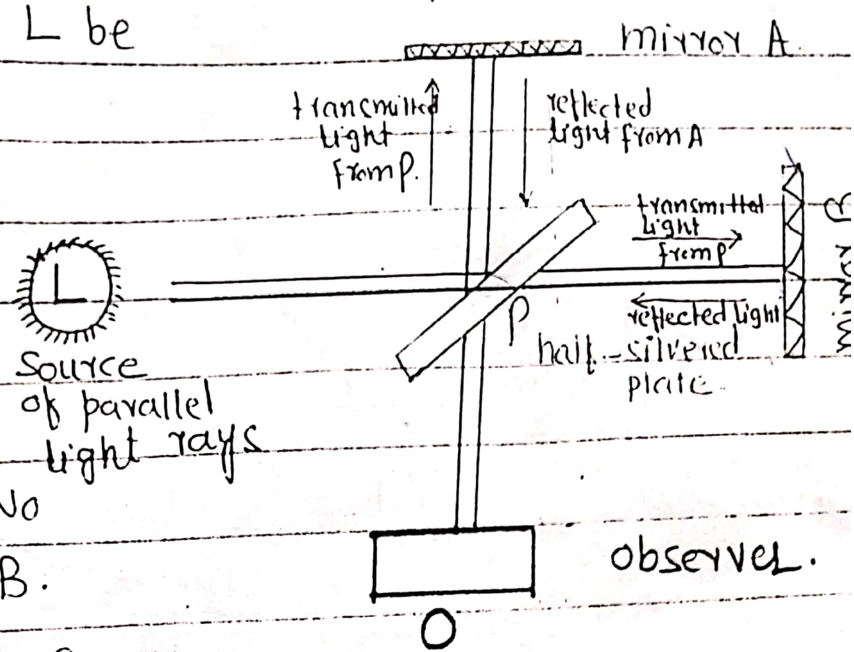
## Non-inertial frame of reference.

The frame of reference in which Newton's l
are not valid is called non-inertial frame
of reference.

5/2015 (3)

# Michelson-Morley Experiment :-

The Michelson-Morley experiment was an attempt to measure the velocity of ~~earth~~ light through the ether. Let L be the source of parallel light coming from a single source, P be the half silvered plate, two mirrors A and B.




The light from a source at L falls on a half-silvered plate P inclined at 45° to the direction of propagation. The plate P, due to half silvered, divides the beam of light into two parts namely reflected and transmitted beams. The two beams, after reflecting at mirrors A and B reach to an observer O.

Let us assume that earth carrying the appara is moving with velocity $\vec{V}$. When the arm B coincide with the motion of ether it will tak time $t_B$ which is given by

$$t_B = \frac{2D/c}{1-V^2/c^2}$$

and when the arm PA is perpendicular to the motion of ether it takes time $t_A$ which is given by

$$t_A = \frac{2D/c}{\sqrt{1 - V^2/c^2}}$$

Where D is the distance of the half-silvered plate from each of the mirror A and B. The time difference between the two paths is given as

$$\Delta t = t_B - t_A$$

$$= \frac{2D/c}{1 - \frac{V^2}{c^2}} - \frac{2D/c}{\sqrt{1 - V^2/c^2}}$$

$$= \frac{2D}{c}\left\{\frac{1}{1 - V^2/c^2} - \frac{1}{\sqrt{1 - V^2/c^2}}\right\}$$

$$= \frac{2D}{c}\left\{\left(1 - \frac{V^2}{c^2}\right)^{-1} - \left(1 - \frac{V^2}{c^2}\right)^{-1/2}\right\}$$

$$= \frac{2D}{c}\left\{\left(1 + \frac{V^2}{c^2}\right) - \left(1 + \frac{1}{2}\frac{V^2}{c^2}\right)\right\}$$ using binomial theorem

$$= \frac{2D}{c}\left\{\cancel{1} + \frac{V^2}{c^2} - \cancel{1} - \frac{1}{2}\frac{V^2}{c^2}\right\}$$

$$= \frac{\cancel{2}D}{c}\left\{\frac{1}{\cancel{2}} V^2/c^2\right\} = \frac{D}{c}\left\{\frac{V^2}{c^2}\right\}$$

$$\Rightarrow \Delta t = \frac{DV^2}{c^3} \longrightarrow (1)$$

It is to be noted that the path difference

of two light waves related with this time difference is

$$d = c\Delta t \quad \text{——— (2)}$$

and the path difference corresponding to shifting of fringes is

$$d = n\lambda \quad \text{——— (3)}$$

where n is number of shifting of fringes.

Comparing (2) and (3), we get,

$$c\Delta t = n\lambda$$

$$\Rightarrow n = \frac{c\Delta t}{\lambda} = \frac{c}{\lambda}\Delta t$$

$$= \left(\frac{c}{\lambda}\right)\frac{DV^2}{c^3} \quad \text{using (1)}$$

$$\Rightarrow n = \left(D/\lambda\right)\left(V^2/c^2\right) \quad \text{——— (4)}$$

Morley used $D = 10\,m$, $\lambda = 5\times10^{-7}\,m$ and $\vec{V} = 3\times10^4\,m/s$.

$$(4) \Rightarrow n = \left(\frac{10}{5\times10^{-7}}\right)\left(\frac{3\times10^4}{3\times10^8}\right)^2$$

$$= (2\times10^7)(10^{-4})^2 = (2\times10^7)(10^{-8})$$

$$= 2\times10^{-1}$$

$$\Rightarrow n = 0.2 \text{ Fringes.}$$

Where $n = 0.2$ is fringe shift of each path. Therefore, the total shift must be equal to $2\times0.2 = 0.4$ which is of significant magnitude and was expected to be observed,

but no fringe shift was found. It wa[s]
expected that, by calculating $\Delta t$, the
absolute speed of ether could be calc[ulated].
But to everybody is surprized, the exper[iment]
failed to give desired result.

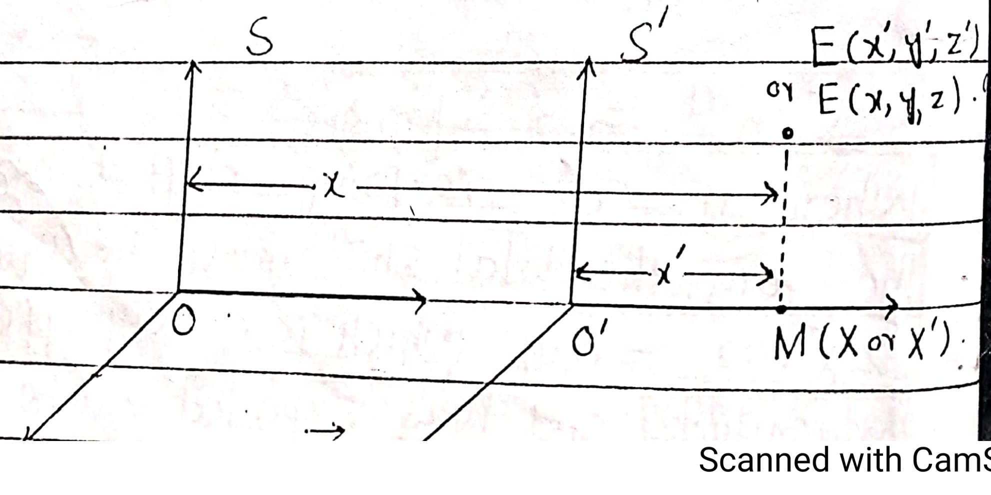
The negative result of Michelson – Mo[rley]
experiment had two consequences:

(i) Ether does not exist in space becau[se]
there will be fringe shift if ether is pr[esent]
in space.

(ii) The speed of light in free space is the
same every where and its value does [not]
depend on velocity of source or observe[r].

## Galilean Transformations Equations:–

Let us consider two frames of reference
S and S′. Let S′ be moving with unifo[rm]
Velocity $\vec{V}$ w.r.t. S. For convenience we
take the motion along one of the coor[dinate]
axes (say x axis).



Further we assume that the coordinates of an event are $(x, y, z)$ at $t$ as observed by an observer in S-frame and say $(x', y', z')$ at $t'$ as observed by an observer in $S'$-frame.

We want to find the relation between $t, x, y, z$ and $t', x', y', z'$. We note that OM and O'M are $x$ and $x'$ coordinates of the event, $OO'$ is the distance covered by $S'$-frame in time $t$

i.e. $\overline{OO'} = \vec{V}t$.

Since there is no motion along $y$ or $z$-axis, therefore, $y' = y$, $z' = z$ and in Newtonian mechanics we take $t' = t$.

The Galilean transformations are

$$t' = t$$

$$x' = x - \vec{V}t$$

$$y' = y$$

$$z' = z$$

$\because \overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$

$x = \vec{V}t + x'$

These are classical kinematic transformatic for uniform linear motion with velocity $\vec{V}$.

Diff. these equations w.r.t. $t'$, we get

$$\frac{dt'}{dt'} = \frac{dt}{dt'} \quad \text{(identity)}.$$

$$\frac{dx'}{dt'} = \frac{d}{dt'}(x - \vec{V}t).$$

$$= \frac{d}{dt}(x - \vec{v}t)\frac{dt}{dt'} = \left(\frac{dx}{dt} - \vec{v}\right)(1)$$

$$\Rightarrow u_x' = u_x - \vec{v}$$

(transformation law for x-component velocity of the event).

$$\frac{dy'}{dt'} = \frac{dy'}{dt'} = \frac{dy}{dt}\frac{dt}{dt'} = \frac{dy}{dt}(1)$$

$$\Rightarrow u_y' = u_y$$

(transformation law for y-component of velocity of the event).

$$\frac{dz'}{dt'} = \frac{dz}{dt'} = \frac{dz}{dt}\cdot\frac{dt}{dt'} = \frac{dz}{dt}(1)$$

$$\Rightarrow u_z' = u_z$$

(transformation law for z-component of velocity of the event)

Thus $\vec{u}' = (u_x', u_y', u_z')$

$$= (u_x - \vec{v}, u_y, u_z)$$

Again differentiating, we get,

$$\vec{a}' = (a_x', a_y', a_z')$$

$$= \frac{d}{dt}(u_x', u_y', u_z')\frac{dt}{dt'}$$

$$= \left(\frac{du_x'}{dt}, \frac{du_y'}{dt}, \frac{du_z'}{dt}\right)(1).$$

$$= \left(\frac{d(u_x - \vec{v})}{dt}, \frac{d}{dt}u_y, \frac{d}{dt}u_z\right)$$

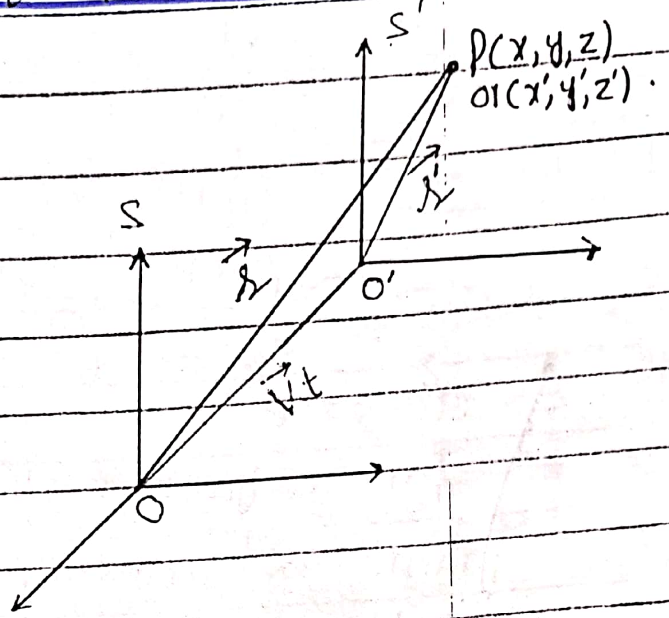
$= (a_x - 0, a_y, a_z) = (a_x, a_y, a_z) = \vec{a}$

$\Rightarrow \vec{a'} = \vec{a}$

This shows that the accelration in two Coordinate system is similar.

## Vectorial form of Galilean transformation :-

Let us assume that the S'-frame moving with uniform velocity $\vec{V}$ relative to S. Then the coordinates as measured by two observers may be connected as follows.




It can be seen in diagram that $\vec{OO'} = \vec{V}t$,

$\vec{OP} = \vec{r}$ and $\vec{O'P} = \vec{r'}$.

$\Rightarrow \vec{OP} = \vec{OO'} + \vec{O'P}$

$\Rightarrow \vec{r} = \vec{V}t + \vec{r'}$

$\Rightarrow \vec{r'} = \vec{r} - \vec{V}t$.

Then the Galilean transformations in vectorial form are

$$\vec{r'} = \vec{r} - \vec{V}t \quad \text{with } t' = t$$

$(\Rightarrow x' = x - V_x t \; ; \; y' = y - V_y t \; ; \; z' = z - V_z t \; ; \; t' = t)$

Diff. w.r.to $t'$, we get,

$$\frac{d\vec{r'}}{dt'} = \frac{d}{dt'}(\vec{r} - \vec{V}t).$$

$$= \frac{d}{dt}(\vec{r} - \vec{v}t)\frac{dt}{dt'}$$

$$= \left(\frac{d\vec{r}}{dt} - \vec{v}\right)(1)$$

$$\Rightarrow \vec{u'} = \vec{u} - \vec{v}$$

Again diff. w.r.t. $t'$, we have

$$\frac{d\vec{u'}}{dt} = \frac{d}{dt'}(\vec{u} - \vec{v}).$$

$$= \frac{d}{dt}(\vec{u} - \vec{v})\frac{dt}{dt'}$$

$$= \left(\frac{d\vec{u}}{dt} - \cancel{\frac{d\vec{v}}{dt}}\right)(1)$$

$$\Rightarrow \vec{a'} = \vec{a}$$

$\Rightarrow$ for a given inertial frame S, any of frame S' moving with uniform spee relative to S is also inertial.

## Inverse Galilean Transformation:-

The inverse of Galilean transformations

$$t' = t,\ x' = x - \vec{v}t,\ y' = y;\ z' = z$$

is $t = t',\ x = x' + \vec{v}t',\ y = y',\ z = z'$

i.e. the inverse transformation is obtain simply by replacing $\vec{v}$ by $-\vec{v}$ and intercha $x, y, z$ by $x', y', z'$.

The inverse transformation of vectori form $\vec{r'} = \vec{r} - \vec{v}t;\ t' = t$

is $\vec{r} = \vec{r'} + \vec{v}t;\ t = t'$.

## Postulates of Special Relativity:-

(i) All the laws of physics are identical in all inertial reference frames.

(ii) The speed of light in free space is constant for all inertial reference frames. Its value in free space is $3 \times 10^8 \, m/s$.

Postulate (i) is called the principle of relativity and (ii) is called principle of constancy of speed of light.

## Lorentz Transformations:- S/2015, S/2018

The set of equations which relates coordinates of a single event in two different reference frames are called Lorentz transformations.

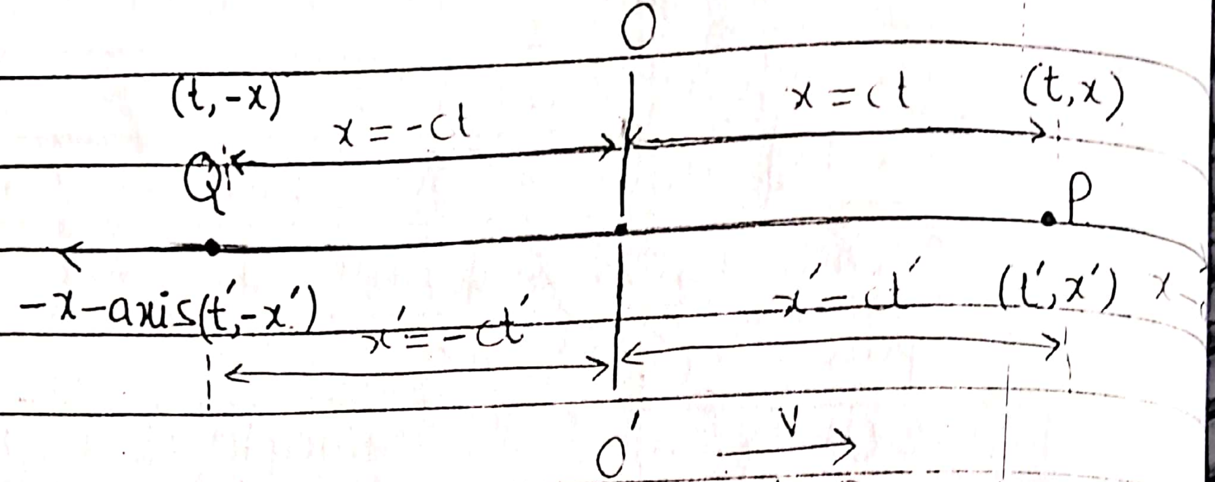
Explanation.

Let us consider two observers O and O' such that the observer O' is moving with speed v in the x-direction relative to O. These two observers coincide at one instant and at that instant they start their clocks. They both send two light signals in the +ve and -ve x-direction.

Since the speed of light is the same for all observers, therefore the signals travel together.

Let O measure time and space by the

Coordinates $(t, x, y, z)$ and $O'$ by $(t', x', y', z')$




Let the signal be at point P in the +ve x-direction and at point Q in the −ve x-direction. Then equations for P and Q, respectively according to O, are

$$P: \; x = ct \quad , \quad Q: \; x = -ct$$

$$\Rightarrow \quad ct - x = 0 \; , \quad ct + x = 0 \quad \text{———} \quad (1)$$

and according to O′, are $\quad$ ($S = vt$)

$$P: \; x' = ct' \quad , \quad Q: \; x' = -ct'$$

$$\Rightarrow \quad ct' - x' = 0 \; , \quad ct' + x' = 0 \quad \text{———} \quad (2)$$

Since P is given by both equations, each implies the other.

So $\quad ct' - x' = \lambda (ct - x) \quad$ ——— (3)

where $\lambda$ is a constant of proportionality.

Similarly, for Q

$$ct' + x' = \mu (ct + x) \quad \text{———} \quad (4)$$

where $\mu$ is a constant of proportionality.

Adding equation (3) and (4), we get,

$$2ct' = (\lambda + \mu) ct + (-\lambda + \mu) x.$$

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$$\Rightarrow ct' = \left(\frac{\lambda+\mu}{2}\right)ct - \left(\frac{\lambda-\mu}{2}\right)x \quad \text{--- (5)}$$

Let us introduce

$$a = \frac{\lambda+\mu}{2} \quad \text{and} \quad \frac{\lambda-\mu}{2} = b$$

Then

$$(5) \Rightarrow t' = at - \frac{b}{c}x \quad \text{--- (6)}$$

Subtracting equation (3) and (4), we get,

$$-2x' = (\lambda-\mu)ct - (\lambda+\mu)x.$$

$$\Rightarrow x' = \left(\frac{\lambda+\mu}{2}\right)x - \left(\frac{\lambda-\mu}{2}\right)ct$$

$$\Rightarrow x' = ax - bct \longrightarrow (7)$$

To determine 'b', the equation for the position x of O' according to O is

$$x = vt$$

and according to O' is $x' = 0$.

Substituting these values in equation (7), we have,

$$0 = avt - bct$$

$$\Rightarrow avt = bct$$

Since this equation holds for all t, so

$$b = \frac{av}{c}$$

Thus equations (6) and (7) become

$$(6) \Rightarrow t' = at - \frac{av}{c^2}x$$

$$\Rightarrow t' = a\left(t - \frac{v}{c^2}x\right) \quad \text{--- (8)}$$

and (7) $\Rightarrow$ $x' = ax - \frac{av}{c} ct$

$\Rightarrow x' = a(x - vt)$ ——— (9)

To determine 'a' we use the initial condi at $t=0$ and $t'=0$, for the two systems

Let us define as $x = x_0$ at $t=0$ and $x' = x_0'$ at $t'=0$.

At $t=0$, (9) $\Rightarrow$ $x' = a(x_0 - 0)$

$\Rightarrow \frac{x'}{x_0} = a$ ——— (10)

At $t'=0$ (9) $\Rightarrow$ $x_0' = a(x - vt)$. ——— (11)

But equation (8) gives at $t'=0$.

$0 = a(t - \frac{v}{c^2}x)$ $\quad \because a \neq 0$.

$\Rightarrow t - \frac{v}{c^2}x = 0$

$\Rightarrow t = \frac{v}{c^2}x$

Putting the value of 't' in equation (11), we get,

$x_0' = a(x - \frac{v^2}{c^2}x) = a(1 - v^2/c^2)x$.

$\Rightarrow \frac{x}{x_0'} = \frac{1}{a}(1 - v^2/c^2)^{-1}$ ——— (12)

By the virtue of equivalence postulate

$\frac{x'}{x_0} = \frac{x}{x_0'}$

(10) and (12) $\Rightarrow$ $a = \frac{1}{a}(1 - v^2/c^2)^{-1}$.

$\Rightarrow a^2 = (1 - v^2/c^2)^{-1}$

$\Rightarrow a = (1 - v^2/c^2)^{-1/2}$

$$\Rightarrow a = \frac{1}{\sqrt{1 - v^2/c^2}} \quad (13)$$

The Lorentz transformations are

$$(8) \Rightarrow t' = a(t - \frac{v}{c^2}x) = \frac{t - v/c^2 x}{\sqrt{1 - v^2/c^2}} = \gamma(t - \frac{v}{c^2}x).$$

$$(7) \Rightarrow x' = a(x - vt) = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \gamma(x - vt).$$

$y' = y$ and $z' = z$.

Thus Lorentz transformations are purely kinematic and have no need to appeal to dynamics or electrodynamics.

**Time dilation:-** A/2010, S/2015, S/2018,

Let us consider two events in $S'$ frame occuring at time $t_1'$ and $t_2'$. Then the time interval between the events is $\Delta t' = t_1' - t_2'$.

Let us assume that an observer in the frame $S$ observes the same event at times $t_1$ and $t_2$. We denote this time interval by $\Delta t = t_1 - t_2$.

Then by Lorentz transformation

$$t_1' = \gamma(t_1 - \frac{v}{c^2}x_1), \quad t_2' = \gamma(t_2 - \frac{v}{c^2}x_2)$$

where $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

Thus $\Delta t' = \gamma(t_1 - \frac{v}{c^2}x_1) - \gamma(t_2 - \frac{v}{c^2}x_2)$.

$$= \gamma[(t_1 - t_2) - \frac{V}{c^2}(x_1 - x_2)]$$

$$\Rightarrow \Delta t' = \gamma(\Delta t - \frac{V}{c^2}(x_1 - x_2)) \quad \text{——— (1)}$$

Since the clock remains at same position

So $x_1 = x_2$.

$$(1) \Rightarrow \Delta t' = \gamma \Delta t$$

$$\Rightarrow \Delta t' = \frac{\Delta t}{\sqrt{1 - V^2/c^2}}$$

This is known as the time dilation form

Length Contraction:- A/2010, S/2015, S/2018

Let $x_1$ and $x_2$ be end points of the ro

placed in S-frame then the length of

in S-frame is

$$\Delta x = x_1 - x_2 \quad \text{——— (1)}$$

whereas the length of the rod in S'-f

appear to be

$$\Delta x' = x_1' - x_2' \quad \text{——— (2)}$$

Since coordinates $x_1', x_2'$ are measured a

the same time, therefore, $t_1' = t_2'$.

So by Lorentz transformation

$$\gamma(t_1 - \frac{V}{c^2}x_1) = \gamma(t_2 - \frac{V}{c^2}x_2)$$

$$\Rightarrow t_1 - \frac{V}{c^2}x_1 = t_2 - \frac{V}{c^2}x_2$$

$$\Rightarrow t_1 - t_2 = \frac{V}{c^2}x_1 - \frac{V}{c^2}x_2$$

$$\Rightarrow t_1 - t_2 = \frac{V}{c^2}(x_1 - x_2)$$

$$\Rightarrow t_1 - t_2 = \frac{V}{c^2}\Delta x \quad \text{———} \quad (3) \quad \text{using } (1)$$

Again using the Lorentz transformations

$$x_1' = \gamma(x_1 - Vt_1), \quad x_2' = \gamma(x_2 - Vt_2)$$

$$(2) \Rightarrow \Delta x' = \gamma(x_1 - Vt_1) - \gamma(x_2 - Vt_2)$$

$$= \gamma[(x_1 - x_2) - V(t_1 - t_2)] \quad \text{———} \quad (4)$$

Subsituting equation (3) in (4), we get

$$\Delta x' = \gamma[\Delta x - \frac{V^2}{c^2}\Delta x]$$

$$= \gamma(1 - V^2/c^2)\Delta x = \gamma\gamma^{-2}\Delta x$$

$$= \frac{\Delta x}{\gamma}$$

$$\Rightarrow \Delta x' = \Delta x\sqrt{1 - V^2/c^2} \qquad \because \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$$

This is called the Lorentz-length Contraction.

Relativity of Simultaneity :-

Consider two events that appear simultaneous to an observer O in S-frame, i.e. one occurs at $x_1$ and the other at $x_2$ at the same time, or in other words $t_1 = t_2$.

The two events would be simultaneous according to O' if $t_1' = t_2'$ but this is not possible infact

$$t_1' - t_2' = \gamma[(t_1 - t_2) - \frac{V}{c^2}(x_1 - x_2)]$$

$= \gamma(0 + V/c^2 (x_2 - x_1))$

$\Rightarrow t_1' - t_2' = \gamma V/c^2 (x_2 - x_1) \neq 0$

$\Rightarrow t_1' \neq t_2'.$

Thus the events do not appear simulta according to O'. Hence simultaneity is relative.

## Velocity Addition Formulae.

By Lorentz transformation

$t' = \gamma(t - V/c^2 x)$

$x' = \gamma(x - vt)$ ; $y' = y$ ; $z' = z$

$\Rightarrow dt' = \gamma(dt - \frac{v}{c^2} dx)$

$dx' = \gamma(dx - vdt)$ ; $dy' = dy$ ; $dz' = dz$

By definition of the speed of any object the x, y and z directions according to

$u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt}.$

Thus $\frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - V/c^2 dx)} = \frac{dx - vdt}{dt - V/c^2 dx}$

$\Rightarrow u_x' = \frac{dx/dt - v}{1 - \frac{v}{c^2} dx/dt} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}.$

$\frac{dy'}{dt'} = \frac{dy}{\gamma(dt - \frac{v}{c^2} dx)} = \frac{dy/dt}{\gamma(1 - \frac{v}{c^2}\frac{dx}{dt})}$

$\Rightarrow u_y' = \frac{u_y}{\gamma(1 - vu_x/c^2)}$

and $\frac{dz'}{dt'} = \frac{dz}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{dz/dt}{\gamma(1 - \frac{v}{c^2}dx/dt)}$

$\Rightarrow u_z' = \frac{u_z}{\gamma(1 - vu_x/c^2)}$.

These are the required velocity addition formulae.

## Relativistic Components of Accelration :-

We can find the relativistic Components of accelration by defining

$$a_x = \frac{du_x}{dt}, \quad a_y = \frac{du_y}{dt}, \quad a_z = \frac{du_z}{dt}.$$

and $a_x' = \frac{du_x'}{dt'}, \quad a_y' = \frac{du_y'}{dt'}, \quad a_z' = \frac{du_z'}{dt'}$

where $a_x' = \frac{d}{dt'}\left[\frac{u_x - v}{1 - \frac{vu_x}{c^2}}\right] = \frac{d}{dt}\left[\frac{u_x - v}{1 - vu_x/c^2}\right]\frac{dt}{dt'}$

where $\frac{dt'}{dt} = \gamma\left(\frac{dt}{dt} - \frac{v}{c^2}\frac{dx}{dt}\right)$

$$= \gamma(1 - vu_x/c^2)$$

so $a_x' = \frac{(1 - vu_x/c^2)(a_x - 0) - (u_x - v)(-va_x/c^2)}{(1 - \frac{vu_x}{c^2})^2 \cdot \gamma(1}$

$$= \frac{a_x(1 - vu_x/c^2) + v/c^2(u_x - v)a_x}{\gamma(1 - vu_x/c^2)^3}$$

$$= \frac{a_x - \cancel{vu_x a_x/c^2} + \cancel{a_x vu_x/c^2} - v^2 a_x/c^2}{\gamma(1 - vu_x/c^2)^3}$$

$$= \frac{a_x(1 - v^2/c^2)}{\gamma(1 - vu_x/c^2)^3} = \frac{a_x\gamma^{-2}}{\gamma(1 - vu_x/c^2)^3}$$

$$= \frac{a_x \gamma^{-3}}{(1 - vu_x/c^2)^3} = \left( \frac{\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} \right)^3 a_x .$$

$$a_y' = \frac{du_y'}{dt'} = \frac{d}{dt'} \left[ \frac{u_y}{\gamma(1 - vu_x/c^2)} \right]$$

$$= \frac{d}{dt} \left[ \frac{u_y}{\gamma(1 - vu_x/c^2)} \right] \frac{dt}{dt'}$$

$$= \frac{(1 - vu_x/c^2) a_y - u_y(-va_x/c^2)}{\gamma^2 (1 - vu_x/c^2)^2} \cdot \frac{1}{\gamma(1 - vu_x/c^2)}$$

$$= \frac{a_y - vu_x a_y/c^2 + vu_y a_x/c^2}{\gamma^3 (1 - vu_x/c^2)^3}$$

$$= \frac{a_y - v/c^2 (u_x a_y - u_y a_x)}{\gamma^3 (1 - vu_x/c^2)}$$

~~a~~ ~~d~~ Similarly,

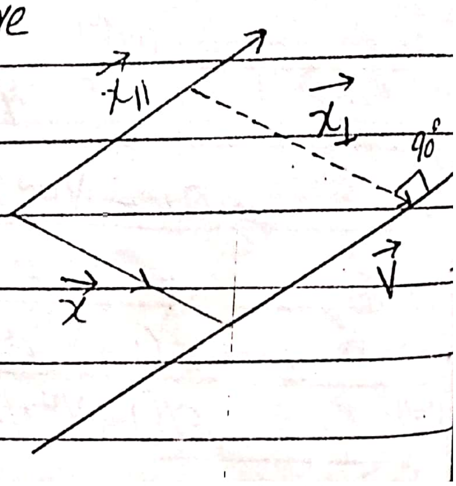
$$a_z' = \frac{a_z - v/c^2 (u_x a_z - u_z a_x)}{\gamma^3 (1 - vu_x/c^2)}.$$

## General Lorentz Transformations. A/2018

Three dimensional Lorentz Transformations.

Let us assume that $S$ and $S'$ are two inertial frame of references where $S'$ is moving with velocity $\vec{v}$ along arbitrary direction.

Let us assume that $\vec{x}$ be the position



vector of the corresponding event E and we split the p.v. $\vec{x}$ into two parts, one along the direction of motion $\vec{x}_{\parallel}$ and the other perpendicular to it, $\vec{x}_{\perp}$, as shown in the Fig.

$$\vec{x} = \vec{x}_{\parallel} + \vec{x}_{\perp} \quad \text{--- (1)}$$

It is clear that there is no effect of uniform motion on $\vec{x}_{\perp}$,

i.e. $$\vec{x}'_{\perp} = \vec{x}_{\perp} \quad \text{--- (2)}$$

However $\vec{x}_{\parallel}$ must transform as the x-coordinate transforms

$$\vec{x}'_{\parallel} = \gamma(\vec{x}_{\parallel} - \vec{v}t) \quad \text{--- (3)}$$

where $\gamma = [1 - \vec{v}\cdot\vec{v}/c^2]^{-1/2}$. --- (4)

Since $|\vec{x}_{\parallel}| = \vec{x}\cdot\hat{v}$, so

$$\vec{x}_{\parallel} = (\vec{x}\cdot\hat{v})\,\hat{v} = \left(\vec{x}\cdot\frac{\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}$$

$$\Rightarrow \vec{x}_{\parallel} = \frac{(\vec{x}\cdot\vec{v})\,\vec{v}}{\vec{v}\cdot\vec{v}} \quad \text{--- (5)}$$

From equation (1), we have

$$\vec{x}_{\perp} = \vec{x} - \vec{x}_{\parallel}$$

$$= \vec{x} - \frac{(\vec{x}\cdot\vec{v})\,\vec{v}}{\vec{v}\cdot\vec{v}} \quad \text{--- (6) :: using (5)}$$

(2) $\Rightarrow$ $$\vec{x}'_{\perp} = \vec{x} - \frac{(\vec{x}\cdot\vec{v})\,\vec{v}}{\vec{v}\cdot\vec{v}} \quad \text{--- (7)}$$

(3) $\Rightarrow$ $$\vec{x}'_{\parallel} = \gamma\left[\frac{(\vec{x}\cdot\vec{v})\,\vec{v}}{\vec{v}\cdot\vec{v}} - \vec{v}t\right] \quad \text{--- (8)}$$

Adding ⑦ and ⑧, we get

$$\vec{x}' = \vec{x}'_{\parallel} + \vec{x}'_{\perp}$$

$$= \gamma\left[\frac{(\vec{x}\cdot\vec{v})\vec{v}}{\vec{v}\cdot\vec{v}} - \vec{v}t\right] + \vec{x} - \frac{(\vec{x}\cdot\vec{v})\vec{v}}{\vec{v}\cdot\vec{v}}$$

$$= \vec{x} + \frac{(\vec{x}\cdot\vec{v})\vec{v}}{\vec{v}\cdot\vec{v}}(\gamma-1) - \gamma\vec{v}t.$$

$$\Rightarrow \vec{x}' = \vec{x} + \vec{v}\left[\frac{(\vec{x}\cdot\vec{v})}{\vec{v}\cdot\vec{v}}(\gamma-1) - \gamma t\right]$$

and $t' = \gamma(t - \vec{v}\cdot\vec{x}/c^2)$

These are called generalized Lorentz transformations.

Time dilation. A/2018

Same derivation as last article page ⑮.

But only change $v, x$ by $\vec{v}, \vec{x}$.

Length Contraction. A/2018

Let $\vec{x}_1$ and $\vec{x}_2$ be the positions ~~vector~~ of end points of a rod placed in S-frame, then the length of the rod in S-frame

$$\vec{l} = \vec{x}_1 - \vec{x}_2 \quad \text{———} \quad (1)$$

whereas the length of the rod in S'-fr appears to be

$$\vec{l}' = \vec{x}'_1 - \vec{x}'_2 \quad \text{———} \quad (2)$$

The general L. transformations for temp coordinates are

$$t' = \gamma(t - \vec{v}\cdot\vec{x}/c^2)$$

$$\Rightarrow t'_1 = \gamma(t_1 - \vec{v}\cdot\vec{x}_1/c^2), \quad t'_2 = \gamma(t_2 - \vec{v}\cdot\vec{x}_2/c^2)$$

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Since the coordinates $\vec{x}_1', \vec{x}_2'$ are measured at the same time, therefore, $t_1' = t_2'$.

$$\Rightarrow \gamma(t_1 - \vec{V}\cdot\vec{x}_1/c^2) = \gamma(t_2 - \vec{V}\cdot\vec{x}_2/c^2)$$

$$\Rightarrow t_1 - \vec{V}\cdot\vec{x}_1/c^2 = t_2 - \vec{V}\cdot\vec{x}_2/c^2$$

$$\Rightarrow t_1 - t_2 = \vec{V}\cdot\vec{x}_1/c^2 - \vec{V}\cdot\vec{x}_2/c^2$$

$$= \frac{\vec{V}\cdot(\vec{x}_1 - \vec{x}_2)}{c^2}$$

$$\Rightarrow t_1 - t_2 = \frac{\vec{V}\cdot\vec{l}}{c^2} \quad (3) \quad \because \text{using } (1)$$

The L. transformations for spatial coordinates are

$$\vec{x}' = \vec{x} + \vec{V}\left[\frac{(\vec{x}\cdot\vec{V})}{\vec{V}\cdot\vec{V}}(\gamma-1) - \gamma t\right]$$

$$\Rightarrow \vec{x}_1' = \vec{x}_1 + \vec{V}\left[\frac{(\vec{x}_1\cdot\vec{V})}{\vec{V}\cdot\vec{V}}(\gamma-1) - \gamma t_1\right]$$

$$\vec{x}_2' = \vec{x}_2 + \vec{V}\left[\frac{(\vec{x}_2\cdot\vec{V})}{\vec{V}\cdot\vec{V}}(\gamma-1) - \gamma t_2\right].$$

Now equation (2) becomes

$$\vec{l}' = (\vec{x}_1 - \vec{x}_2) + \vec{V}\left[\frac{(\vec{x}_1 - \vec{x}_2)\cdot\vec{V}}{\vec{V}\cdot\vec{V}}(\gamma-1) - \gamma(t_1 - t_2)\right]$$

using (1) and (3) in above equation, we get.

$$\vec{l}' = \vec{l} + \vec{V}\left[\frac{\vec{l}\cdot\vec{V}}{\vec{V}\cdot\vec{V}}(\gamma-1) - \gamma\frac{\vec{V}\cdot\vec{l}}{c^2}\right]$$

$$= \vec{l} + \frac{\vec{V}(\vec{V}\cdot\vec{l})}{\vec{V}\cdot\vec{V}}\left[(\gamma-1) - \gamma\frac{\vec{V}\cdot\vec{V}}{c^2}\right] = \vec{l} + \frac{\vec{V}(\vec{V}\cdot\vec{l})}{\vec{V}\cdot\vec{V}}[\gamma(1$$

## Velocity Addition Formulae:

The G. L. transformations are

$$t' = \gamma(t - \vec{v}\cdot\vec{x}/c^2) \quad \text{--- (1)}$$

$$\Rightarrow \frac{dt'}{dt} = \gamma\left(\frac{dt}{dt} - \frac{\vec{v}}{c^2}\frac{d\vec{x}}{dt}\right)$$

$$= \gamma\left(1 - \frac{\vec{v}\cdot\vec{u}}{c^2}\right)$$ where $u$ is the velo[city] of an event.

and $\vec{x}' = \vec{x} + \vec{v}\left[\frac{(\vec{x}\cdot\vec{v})}{\vec{v}\cdot\vec{v}}(\gamma - 1) - \gamma t\right]$

$$\Rightarrow \vec{u}' = \frac{d\vec{x}'}{dt'} = \frac{d}{dt}\left[\vec{x} + \vec{v}\left[\frac{(\vec{x}\cdot\vec{v})}{\vec{v}\cdot\vec{v}}(\gamma-1) - \gamma t\right]\right]$$

$$= \left(\vec{u} + \vec{v}\left[\frac{(\vec{u}\cdot\vec{v})}{\vec{v}\cdot\vec{v}}(\gamma-1) - \gamma\right]\right)\frac{1}{\gamma(1 - \vec{v}\cdot\vec{u}/c^2)}$$

$$= \frac{\vec{u} + \vec{v}\left[(\vec{u}\cdot\vec{v})/\vec{v}\cdot\vec{v}\,(\gamma-1) - \gamma\right]}{\gamma\left(1 - \frac{\vec{v}\cdot\vec{u}}{c^2}\right)}.$$

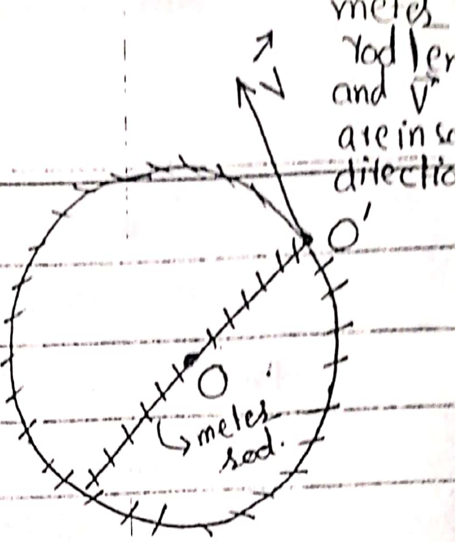
which is the relation for relativistic la[w] of addition of velocities.

**Question.**

Show that in uniform circular motion the perimeter of the circular path appears t[o] be shortened by $\gamma^{-1}$ factor.

**Solution.**

Let us consider an observer in uniform circular motion, which is obviously an




accelrated motion.
We assume ~~that~~ the meter rods constitutes its diameter and the circumference.
Let us further assume that $P$ = perimeter of the circle observed by an observer $O$ and $d$ = diameter of the circle observed by an observer $O$. The ratio of the perimeter to its diameter is $P/d$.
Let $P'$ and $d'$ be the perimeter and the diameter as observed by the observer $O'$.
The rods making up the circle itself, can be assumed in the same direction as that of the instantaneous motion of $O'$.
Thus the lengths of the rod making up the circle are shortened by $\gamma^{-1}$ factor.
Thus $P'$ = perimeter observed by $O'$ is shorten by the factor $\gamma^{-1}$.
The motion of $O'$ w.r.t. the rods making up the diameter is such that $\vec{v}$ is ~~such~~ $\perp$ to diameter $d$, i.e. $\vec{v} \cdot \vec{d} = 0$.
$\Rightarrow d' = d$ (i.e. unaffected length of diameter)
$\Rightarrow$ The ratio of perimeter $P'$ and $d'$ $= P'/d'$

$$= \frac{\gamma^{-1} P}{d} = \frac{1}{\gamma}\left(\frac{P}{d}\right) = \frac{1}{\gamma}\pi = \gamma^{-1}\pi.$$

Thus $\Lambda' = \Lambda^{-1}\Lambda^{t} \neq \Lambda$.
i.e. the geometry in this case is non-Euc

Positive definite metric.
A metric $ds^2 = g_{ij}\,dx^i dx^j$ is said to b
positive definite if $ds^2 \geq 0$.
Indefinite metric.
A metric $ds^2 = g_{ij}\,dx^i dx^j$ is said to b
indefinite if $ds^2 < 0$.

Question. S/2018
Show that line element is invarient
Lorentz transformations.
Solution.
The line element in Minkoski coordinat
is given by

$$ds^2 = g_{ij}\,dx^i dx^j$$
$$= c^2dt^2 - dx^2 - dy^2 - dz^2$$
$$\Rightarrow g_{00} = 1;\ g_{11} = -1;\ g_{22} = -1;\ g_{33} = -1$$
$$(x^0, x^1, x^2, x^3) = (ct, x, y, z) \quad ; T = ct = x^0$$
$$g_{ij} = 0 \quad \forall\ i \neq j. \qquad \cancel{dT = cdt}$$

The L.Ts in Minkoski coordinate are
given by

$$t' = \gamma(t - vx/c^2)$$
$$\Rightarrow ct' = \gamma(ct - vx/c)$$
$$\Rightarrow x^{0'} = \gamma(x^0 - vx^1/c)$$
$$x' = \gamma(x - vt).$$

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Method:
Step I, line element in Minkowski space
Step II, L.Ts in Minkowski space
Step III, $g_{ij} =$ —— Inverse L.Ts
Step IV: $g'_{ij} = ?$

$$\Rightarrow x' = \gamma\left(x - \frac{V(ct)}{c}\right)$$

$$\Rightarrow x^{1'} = \gamma(x' - Vx^0/c)$$

$$y' = y$$

$$\Rightarrow x^{2'} = x^2$$ (x-two)

$$z' = z \quad \Rightarrow \quad x^{3'} = x^3.$$ {x-three}

Thus in Minkoski coordinates L.Ts is

$$x^{0'} = \gamma(x^0 - Vx^1/c)$$

$$= \gamma x^0 - \gamma Vx^1/c + 0x^2 + 0x^3.$$

$$x^{1'} = \gamma(x^1 - Vx^0/c)$$

$$= \gamma x^1 - \gamma Vx^0/c + 0x^2 + 0x^3.$$

$$x^{2'} = 0x^0 + 0x^1 + x^2 + 0x^3.$$

$$x^{3'} = 0x^0 + 0x^1 + 0x^2 + x^3.$$

$$\Rightarrow \begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma V/c & 0 & 0 \\ -\gamma V/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Now

$$g'_{ij} = \frac{\partial x^\alpha}{\partial x^{i'}} \cdot \frac{\partial x^\beta}{\partial x^{j'}} g_{\alpha\beta}$$ for which, we have

$$\boxed{g_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^{'\mu}} \frac{\partial x^\beta}{\partial x^{'\nu}}}$$

The inverse L.Ts are given by

$$x^0 = \gamma(x^{0'} + Vx^{1'}/c)$$

$$x^1 = \gamma(x^{1'} + Vx^{0'}/c)$$

$$x^2 = x^{2'}$$

$$x^3 = x^{3'}.$$

The Jacobian matrix is given by

(28)

$$\left(\frac{\partial x^{\alpha}}{\partial x^{i'}}\right) = \begin{pmatrix} \partial x^0/\partial x^{0'} & \partial x^0/\partial x^{1'} & \partial x^0/\partial x^{2'} & \partial x \\ \partial x^1/\partial x^{0'} & \partial x^1/\partial x^{1'} & \partial x^1/\partial x^{2'} & \partial x \\ \partial x^2/\partial x^{0'} & \partial x^2/\partial x^{1'} & \partial x^2/\partial x^{2'} & \partial x \\ \partial x^3/\partial x^{0'} & \partial x^3/\partial x^{1'} & \partial x^3/\partial x^{2'} & \partial x \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & \gamma V/c & 0 & 0 \\ \gamma V/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now

$$g'_{ij} = \frac{\partial x^{\alpha}}{\partial x^{i'}} \frac{\partial x^{\beta}}{\partial x^{j'}} g_{\alpha\beta}.$$

$$g'_{00} = \frac{\partial x^{\alpha}}{\partial x^{0'}} \frac{\partial x^{\beta}}{\partial x^{0'}} g_{\alpha\beta}.$$

$$= \frac{\partial x^0}{\partial x^{0'}} \frac{\partial x^0}{\partial x^{0'}} g_{00} + \frac{\partial x^1}{\partial x^{0'}} \frac{\partial x^1}{\partial x^{0'}} g_{11}$$

$$+ \frac{\partial x^2}{\partial x^{0'}} \frac{\partial x^2}{\partial x^{0'}} g_{22} + \frac{\partial x^3}{\partial x^{0'}} \frac{\partial x^3}{\partial x^{0'}} g_{33}$$

$$= \left(\frac{\partial x^0}{\partial x^{0'}}\right)^2 g_{00} + \left(\frac{\partial x^1}{\partial x^{0'}}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial x^{0'}}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial x^{0'}}\right)^2 g_{33}$$

$$= \gamma^2(1) + (\gamma V/c)^2(-1) + 0 + 0$$

$$= \gamma^2 - \gamma^2 V^2/c^2$$

$$= \gamma^2(1 - V^2/c^2) = \gamma^2(\gamma^{-2}) = 1$$

$$\Rightarrow g'_{00} = 1.$$

$$g'_{11} = \frac{\partial x^{\alpha}}{\partial x^{1'}} \frac{\partial x^{\beta}}{\partial x^{1'}} g_{\alpha\beta} .$$

$$= \left(\frac{\partial x^0}{\partial x^{1'}}\right)^2 g_{00} + \left(\frac{\partial x^1}{\partial x^{1'}}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial x^{1'}}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial x^{1'}}\right)^2 g_{33}$$

$$= (\gamma V/c)^2 (1) + (\gamma)^2 (-1) + 0 + 0$$

$$= \gamma^2 V^2/c^2 - \gamma^2$$

$$= -\gamma^2 (1 - V^2/c^2) = -\gamma^2 (\gamma^{-2}) = -1$$

$$\Rightarrow g'_{11} = -1 .$$

$$g'_{22} = \left(\frac{\partial x^0}{\partial x^{2'}}\right)^2 g_{00} + \left(\frac{\partial x^1}{\partial x^{2'}}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial x^{2'}}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial x^{2'}}\right)^2 g_{33}$$

$$= 0 + 0 + (1)^2 (-1) + 0 = -1$$

$$\Rightarrow g'_{22} = -1$$

$$g'_{33} = \left(\frac{\partial x^0}{\partial x^{3'}}\right)^2 g_{00} + \left(\frac{\partial x^1}{\partial x^{3'}}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial x^{3'}}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial x^{3'}}\right)^2 g_{33}$$

$$= 0 + 0 + 0 + (1)^2 (-1) = -1$$

$$\Rightarrow g'_{33} = -1 .$$

Therefore,

$$ds^2 = g_{ij} \, dx^i dx^j$$

$$= g'_{ij} \, dx^{i'} dx^{j'}$$

$$= g'_{00} (dx^{0'})^2 + g'_{11} (dx^{1'})^2 + g'_{22} (dx^{2'})^2 + g'_{33} (dx^{3'})^2$$

$$= (1)(c^2 dt'^2) - dx'^2 - dy'^2 - dz'^2$$

$$= c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$$

$$\Rightarrow ds^2 = ds'^2 .$$

Hence proved.

The L-Transformations in 4-vectors.

The coordinate $(ct, x, y, z) = (x^0, x^1, x^2, x^3)$

$$x'^i = A^i_j x^j$$

$$\Rightarrow x'^0 = A^0_j x^j = A^0_0 x^0 + A^0_1 x^1 + A^0_2 x^2 + A^0_3 x^3$$

$$x'^1 = A^1_j x^j = A^1_0 x^0 + A^1_1 x^1 + A^1_2 x^2 + A^1_3 x^3$$

$$x'^2 = A^2_j x^j = A^2_0 x^0 + A^2_1 x^1 + A^2_2 x^2 + A^2_3 x^3$$

$$x'^3 = A^3_j x^j = A^3_0 x^0 + A^3_1 x^1 + A^3_2 x^2 + A^3_3 x^3$$

$$\Rightarrow \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} A^0_0 & A^0_1 & A^0_2 & A^0_3 \\ A^1_0 & A^1_1 & A^1_2 & A^1_3 \\ A^2_0 & A^2_1 & A^2_2 & A^2_3 \\ A^3_0 & A^3_1 & A^3_2 & A^3_3 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

The L.Ts can be written as $\because ct = x^0$

$$ct' = \gamma(ct - vx/c)$$

$$\Rightarrow x'^0 = \gamma\left(x^0 - \frac{vx^1}{c}\right) = \gamma x^0 - \gamma\frac{vx^1}{c} + 0x^2 + 0x^3$$

$$x'^1 = \gamma(x - vt)$$

$$= \gamma\left(x - \frac{vct}{c}\right) = \gamma(x^1 - vx^0/c)$$

$$= -\gamma\frac{v}{c}x^0 + \gamma x^1 + 0x^2 + 0x^3$$

$$y' = y \Rightarrow x^{2'} = x^2 = 0x^0 + 0x^1 + 0x^2 + 0x^3$$

$$z' = z \Rightarrow x^{3'} = x^3 = 0x^0 + 0x^1 + 0x^2 + x^3$$

The linear transformation $x'^i = A^i_j x^j$ are L.Ts if $A^i_j$, the matrix of transformation, is given by

$$A^i_j = \begin{pmatrix} A^0_0 & A^0_1 & A^0_2 & A^0_3 \\ A^1_0 & A^1_1 & A^1_2 & A^1_3 \\ A^2_0 & A^2_1 & A^2_2 & A^2_3 \\ A^3_0 & A^3_1 & A^3_2 & A^3_3 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma V/c & 0 & 0 \\ -\gamma V/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(1) Thus in case of one spatial dimension, the active part of transformation is given by

$$x'^{i} = A^{i}_{j} x^{j} \quad ; \quad i,j = 0,1.$$

$$\begin{pmatrix} x'^{0} \\ x'^{1} \end{pmatrix} = \begin{pmatrix} A^{0}_{0} & A^{0}_{1} \\ A^{1}_{0} & A^{1}_{1} \end{pmatrix} \begin{pmatrix} x^{0} \\ x^{1} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} T' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma V/c \\ -\gamma V/c & \gamma \end{pmatrix} \begin{pmatrix} T \\ x \end{pmatrix} \qquad \because x^{0} = ct = T$$

$$T' = \gamma T - \gamma \frac{Vx}{c} = \gamma\left(T - Vx/c\right)$$

$$x' = -\frac{\gamma V}{c} T + \gamma x = \gamma\left(x - \frac{VT}{c}\right) \qquad \because T = ct$$

Let us introduce $\beta = V/c$ then $\gamma = (1 - V^2/c^2)^{-1/2}$.

$$\Rightarrow \gamma = (1-\beta^2)^{-1/2}$$

$$\Rightarrow \begin{pmatrix} T' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} T \\ x \end{pmatrix}.$$

where $A^{i}_{j} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} = B$ (say).

$$|A^{i}_{j}| = \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = \gamma^2\gamma^{-2} = +1 \neq 0$$

$$B^{-1} = (A^{i}_{j})^{-1} = \frac{\operatorname{Adj} B}{|B|} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

Thus the inverse Lorentz transformations are

given by Simply interchanging the primed and unprimed Coordinates and $\beta$ by $-\beta$.

② L.Ts in terms of hyperbolic function.

It is to be noted that

$|A^i_j| = 1$ obeys the same relation as that $\cosh^2\theta - \sinh^2\theta = 1$.

i.e. if we replace $\gamma = (1 - v^2/c^2)^{-1/2} = (1-\beta^2)^{-1/2} = \cosh\theta$

and $\gamma\beta = (1-\beta^2)^{-1/2}\beta = \sinh\theta$ then

$$\gamma^2 - \gamma^2\beta^2 = \cosh^2\theta - \sinh^2\theta = 1.$$

The transformation matrix may be written as

$$A^i_j = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x^{0'} \\ x^{1'} \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

$$\Rightarrow x^{0'} = \cosh\theta\, x^0 - \sinh\theta\, x^1$$

$$x^{1'} = -\sinh\theta\, x^0 + \cosh\theta\, x^1$$

$$\Rightarrow \left.\begin{aligned} T' &= T\cosh\theta - x\sinh\theta \\ x' &= -T\sinh\theta + x\cosh\theta \end{aligned}\right\} \text{——— (1)}$$

$$\text{Since } \left.\begin{aligned} \cos(i\theta) &= \cosh\theta \\ \sin(i\theta) &= i\sinh\theta \end{aligned}\right\} \text{——— (2)}$$

$$(1) \Rightarrow \left.\begin{aligned} T' &= T\cosh\theta + ix(i\sinh\theta) \\ x' &= iT(i\sinh\theta) + x\cosh\theta \end{aligned}\right\} \text{——— (3)}$$

② in ③ $\Rightarrow$ $T' = T\cos(i\theta) + ix_i \sin(i\theta)$

$x' = iT\sin(i\theta) + x\cos(i\theta)$.

$$\Rightarrow \begin{pmatrix} T' \\ x' \end{pmatrix} = \begin{pmatrix} \cos(i\theta) & i\sin(i\theta) \\ i\sin(i\theta) & \cos(i\theta) \end{pmatrix} \begin{pmatrix} T \\ x \end{pmatrix}.$$

Thus the time and space appear mixed together, they are no longer separate entities and the L.Ts correspond to a relation through an imaginary angle.

Question.

Show that special L. Transformations form a group under the successive application of transformations.

Solution.

① Closure Property.

The combination of two L. transformations must be a L. transformation.

Let $V_1$ and $V_2$ be the two uniform velocities such that $\beta_1 = V_1/c$ and $\beta_2 = V_2/c$ so that the two L. transformations are given by

$$\begin{pmatrix} T_1' \\ x_1' \end{pmatrix} = \begin{pmatrix} \cosh\theta_1 & -\sinh\theta_1 \\ -\sinh\theta_1 & \cosh\theta_1 \end{pmatrix} \begin{pmatrix} T_1 \\ x_1 \end{pmatrix} = A_1 \begin{pmatrix} T_1 \\ x_1 \end{pmatrix}$$

and

$$\begin{pmatrix} T_2' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cosh\theta_2 & -\sinh\theta_2 \\ -\sinh\theta_2 & \cosh\theta_2 \end{pmatrix} \begin{pmatrix} T_2 \\ x_2 \end{pmatrix} = A_2 \begin{pmatrix} T_2 \\ x_2 \end{pmatrix}$$

The resultant of two transformations can be found as

$$A_1 A_2 = \begin{pmatrix} \cosh\theta_1 & -\sinh\theta_1 \\ -\sinh\theta_1 & \cosh\theta_1 \end{pmatrix} \begin{pmatrix} \cosh\theta_2 & -\sinh\theta_2 \\ -\sinh\theta_2 & \cosh\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\theta_1+\theta_2) & -\sinh(\theta_1+\theta_2) \\ -\sinh(\theta_1+\theta_2) & \cosh(\theta_1+\theta_2) \end{pmatrix}$$

The resultant of two transformations is given

$$\begin{pmatrix} T' \\ x' \end{pmatrix} = A_1 A_2 \begin{pmatrix} T \\ x \end{pmatrix} = \begin{pmatrix} \cosh(\theta_1+\theta_2) & -\sinh(\theta_1+\theta_2) \\ -\sinh(\theta_1+\theta_2) & \cosh(\theta_1+\theta_2) \end{pmatrix} \begin{pmatrix} T \\ x \end{pmatrix}$$

But

$$\begin{pmatrix} T' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} T \\ x \end{pmatrix}$$

$$\Rightarrow \gamma = \cosh\theta = \cosh(\theta_1+\theta_2)$$

$$-\gamma\beta = -\sinh\theta = -\sinh(\theta_1+\theta_2)$$

$$\Rightarrow \beta = \frac{\sinh(\theta_1+\theta_2)}{\gamma} = \frac{\sinh(\theta_1+\theta_2)}{\cosh(\theta_1+\theta_2)}$$

$$= \frac{\sinh\theta}{\cosh\theta} = \tanh\theta = \tanh(\theta_1+\theta_2).$$

$$\Rightarrow \beta = \frac{\tanh\theta_1 + \tanh\theta_2}{1+\tanh\theta_1 \tanh\theta_2} \qquad \because \beta_1 = \tanh\theta_1,\ \beta_2 = \tanh\theta_2$$

$$\Rightarrow \beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

$$\Rightarrow \frac{V}{c} = \frac{V_1/c + V_2/c}{1+V_1/c \cdot V_2/c} \qquad \because \beta = V/c.$$

$$\Rightarrow \frac{1}{c} V = \frac{(V_1+V_2)\,1/c}{1+V_1V_2/c^2}$$

$$\Rightarrow V = \frac{V_1+V_2}{1+V_1V_2/c^2}$$ which is relativistic law of addition of velocities

and the transformation

$$\begin{pmatrix} T' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} T \\ x \end{pmatrix}$$ is itself a L. transformatio

and hence the L. transformations are closed under the successive application of transformation.

② Associative Property.

If $\begin{pmatrix} T \\ x \end{pmatrix} = A_1 \begin{pmatrix} T' \\ x' \end{pmatrix}$ ; $\begin{pmatrix} T' \\ x' \end{pmatrix} = A_2 \begin{pmatrix} T'' \\ x'' \end{pmatrix}$ and $\begin{pmatrix} T'' \\ x'' \end{pmatrix} = A_3 \begin{pmatrix} T''' \\ x''' \end{pmatrix}$

Then $\begin{pmatrix} T \\ x \end{pmatrix} = A_1 A_2 A_3 \begin{pmatrix} T''' \\ x''' \end{pmatrix}$

Now the associativity of matrices multiplication guarantees the associativity of L.Ts.

③ Existence of identity Transformation.

we note that

$$\begin{pmatrix} T \\ x \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} T' \\ x' \end{pmatrix}$$ is an identity mapping

for $V=0$ as follows

$$\gamma = (1 - V^2/c^2)^{-1/2} = 1 \quad ; \quad V=0$$

$$\gamma\beta = (1 - V^2/c^2)^{-1/2} \left(\frac{V}{c}\right) = 0 \quad ; \quad V=0$$

i.e. $T' = T$ ; $x' = x$.

④ Existence of Inverse mapping.

The inverse of each L.Ts can be obtained simply by replacing $\beta$ by $-\beta$.

i.e. Hence a set of L.Ts form a group under the successive application of transformations.

Question: A/2011

Show that General Lorentz Transformation form a group under the successive application of transformations.

Solution.

The General L.Ts are given by

$$t' = \gamma(t - \vec{V}\cdot\vec{x}/c^2) \quad \text{--- (1)}$$

$$\vec{x}' = \vec{x} + \vec{V}\left[\frac{\vec{x}\cdot\vec{V}}{\vec{V}\cdot\vec{V}}(\gamma-1) - \gamma t\right] \quad \text{--- (2)}$$

Let us rewrite these transformations in Comp form and denote $\vec{\beta} = \vec{V}/c = (\beta_1, \beta_2, \beta_3)$ and $ct' = T'$, $ct = T$, $x^1 = x$, $x^2 = y$, $x^3 = z$.

$$(1) \Rightarrow T' = \gamma(ct - \vec{\beta}\cdot\vec{x}) = \gamma(T - \vec{\beta}\cdot\vec{x}) \quad \text{---}$$

$$(2) \Rightarrow \vec{x}' = \vec{x} + c\vec{\beta}\left[\frac{\vec{x}\cdot c\vec{\beta}}{c^2(\vec{\beta}\cdot\vec{\beta})}(\gamma-1) - \frac{\gamma}{c}T\right]$$

$$= \vec{x} + \vec{\beta}\left[\frac{\vec{x}\cdot\vec{\beta}}{\vec{\beta}\cdot\vec{\beta}}(\gamma-1) - \gamma T\right] \quad \text{---}$$

$$(3) \Rightarrow T' = \gamma[T - (\beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3)]$$

$$\Rightarrow T' = \gamma T - \gamma\beta_1 x^1 - \gamma\beta_2 x^2 - \gamma\beta_3 x^3 \quad \text{---}$$

$$(4) \Rightarrow x^{1'} = x^1 + \beta_1\left[\vec{x}\cdot\vec{\beta}(\gamma-1)/\vec{\beta}\cdot\vec{\beta} - \gamma T\right] \quad \text{---}$$

$$x^{2'} = x^2 + \beta_2\left[\vec{x}\cdot\vec{\beta}(\gamma-1)/\vec{\beta}\cdot\vec{\beta} - \gamma T\right] \quad \text{---}$$

$$x^{3'} = x^3 + \beta_3\left[\vec{x}\cdot\vec{\beta}(\gamma-1)/\vec{\beta}\cdot\vec{\beta} - \gamma T\right] \quad \text{---}$$

$$(6) \Rightarrow x^{1'} = \beta_1\left[\frac{x^1\beta_1 + x^2\beta_2 + x^3\beta_3}{\vec{\beta}\cdot\vec{\beta}}(\gamma-1) - \gamma T\right] + x^1$$

$$x^{1'} = -\gamma\beta_1 T + \left[1 + \beta_1^2(\gamma-1)/\vec{\beta}\cdot\vec{\beta}\right]x^1 + \beta_1\beta_2(\gamma-1)/\vec{\beta}\cdot\vec{\beta}\, x^2 + \frac{\beta_1\beta_3}{\vec{\beta}\cdot\vec{\beta}}(\gamma-1)x^3 \quad \text{---} \quad (9)$$

$$(7) \Rightarrow x^{2'} = -\gamma\beta_2 T + \beta_1\beta_2(\gamma-1)/\vec{\beta}\cdot\vec{\beta}\, x^1 + \left[1+\beta_2^2(\gamma-1)/\vec{\beta}\cdot\vec{\beta}\right]x^2 + \beta_2\beta_3(\gamma-1)/\vec{\beta}\cdot\vec{\beta}\, x^3 \quad \text{---} \quad (10)$$

$$(8) \Rightarrow x^{3'} = -\gamma\beta_3 T + \beta_1\beta_3(\gamma-1)/\vec{\beta}\cdot\vec{\beta}\, x^1 + \beta_2\beta_3(\gamma-1)/\vec{\beta}\cdot\vec{\beta}\, x^2 + \left[1+\beta_3^2(\gamma-1)/\vec{\beta}\cdot\vec{\beta}\right]x^3 \quad \text{---} \quad (11)$$

The L.Ts in component form are given by (5) and (9)—(11). They can be written in matrix for

i.e. $\vec{X}' = A\vec{X}$

where

$$\vec{X}' = \begin{pmatrix} T' \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} T \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

and

$$A = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\beta_1^2(\gamma-1)/\beta^2 & \beta_1\beta_2(\gamma-1)/\beta^2 & \beta_1\beta_3(\gamma-1)/\beta^2 \\ -\gamma\beta_2 & \beta_1\beta_2(\gamma-1)/\beta^2 & 1+\beta_2^2(\gamma-1)/\beta^2 & \beta_2\beta_3(\gamma-1)/\beta^2 \\ -\gamma\beta_3 & \beta_1\beta_3(\gamma-1)/\beta^2 & \beta_2\beta_3(\gamma-1)/\beta^2 & 1+\beta_3^2(\gamma-1)/\beta^2 \end{pmatrix}$$

Now we show that General Lorentz transfo form a group.

(1) Closure Property.

Let us consider another L.T with $\vec{\eta} = (\eta_1, \eta_2, \eta_3)$, $\delta = (1 - \vec{w}\cdot\vec{w}/c^2)^{-1/2}$ governed by

the matrix

$$B = \begin{pmatrix} \delta & -\delta\eta_1 & -\delta\eta_2 & -\delta\eta_3 \\ -\delta\eta_1 & 1+\eta_1^2(\delta-1)/\vec{\eta}\cdot\vec{\eta} & \eta_1\eta_2(\delta-1)/\vec{\eta}\cdot\vec{\eta} & \eta_1\eta_3 \ldots \\ -\delta\eta_2 & \eta_1\eta_2(\delta-1)/\vec{\eta}\cdot\vec{\eta} & 1+\eta_2^2(\delta-1)/\vec{\eta}\cdot\vec{\eta} & \eta_2\eta_3 \ldots \\ -\delta\eta_3 & \eta_1\eta_3(\delta-1)/\vec{\eta}\cdot\vec{\eta} & \eta_2\eta_3(\delta-1)/\vec{\eta}\cdot\vec{\eta} & 1+ \ldots \end{pmatrix}$$

$$AB = \begin{pmatrix} \bar{\lambda} & -\bar{\lambda}\varphi_1 & -\bar{\lambda}\varphi_2 & -\bar{\lambda}\varphi_3 \\ -\bar{\lambda}\varphi_1 & 1+\varphi_1^2(\bar{\lambda}-1)/\vec{\varphi}\cdot\vec{\varphi} & \varphi_1\varphi_2(\bar{\lambda}-1)/\vec{\varphi}\cdot\vec{\varphi} & \varphi_1\varphi_3(\bar{\lambda} \ldots \\ -\bar{\lambda}\varphi_2 & \varphi_1\varphi_2(\bar{\lambda}-1)/\vec{\varphi}\cdot\vec{\varphi} & 1+\varphi_2^2(\bar{\lambda}-1)/\vec{\varphi}\cdot\vec{\varphi} & \varphi_2\varphi_3(\bar{\lambda} \ldots \\ -\bar{\lambda}\varphi_3 & \varphi_1\varphi_3(\bar{\lambda}-1)/\vec{\varphi}\cdot\vec{\varphi} & \varphi_1\varphi_3(\bar{\lambda}-1)/\vec{\varphi}\cdot\vec{\varphi} & 1+\varphi_3^2(\bar{\lambda} \ldots \end{pmatrix}$$

where $\bar{\lambda} = \left(1 - \frac{\vec{u}\cdot\vec{u}}{c^2}\right)^{-1/2}$

and $\vec{u}' = \dfrac{\vec{w} + \vec{v}\left[(\vec{w}\cdot\vec{v})\frac{(\bar{\lambda}-1)}{\vec{v}\cdot\vec{v}} - \bar{\lambda}\right]}{\bar{\lambda}\left[1 - \vec{v}\cdot\vec{w}/c^2\right]}$ veloc addit law

The Combination of two L-transformations is itself a L-transformation.

② Associative Property.

Now the associativity of matrices multiplication gaurantees the associativ of L.Ts.

③ Existence of identity transformation

we nothe that $\begin{pmatrix} T \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = A \begin{pmatrix} T' \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix}$ is an identity

transformation for $\gamma = 1$, $\beta = 0$.

$\Rightarrow x^1 = x^{1\prime}$ ; $x^2 = x^{2\prime}$ ; $x^3 = x^{3\prime}$ ; $T = T'$.

(4) Existence of inverse transformation.

The inverse of each L-transformation can be obtained simply by replacing $\beta$ by $-\beta$.

Hence a set of general Lorentz transform form a group under the successive appli. of transformations.

Minkowski's Space.

The 4-dimensional space in which a point represents an event occuring at a certain point in ordinary space and at a certain time is called Minkowski's Space.

Special Cases of Lorentz transformation and Lorentz group. :-

Lorentz group. S/2015, A/2016,

"Lorentz group is the group admitted by Minkowski space time.

i.e. the space time governed by the metric -

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

"A linear transformation in space time continuum is given by

$$x'^i = A^i_j x^j + b^i \quad ; \quad i = 0, 1, 2, 3$$

where $A^i_j$ is the matrix of transformations."

For every Lorentz transformation given the matrix $A^i_j$, there exists an identity element, inverse and satisfies the closure and associative property.

Thus forming a group under successive applications of L.Ts, these transformations admit a group which is called Lorentz group and is denoted by L.

The existence of inverse for any L.T is ensured by the property that

$$(A^i_j)(A^i_j)^{-1} = (A^i_j)(\bar{A}')^j_k = \delta^i_k$$

which is called orthogonality condition.

However, the orthogonality of transformation means that for

$$B = A^i_j \quad \Rightarrow \quad BB^t = I$$

$$\Rightarrow |BB^t| = |I| \quad \Rightarrow \quad |BB^t| = 1$$

$$\Rightarrow |B||B^t| = 1 \quad \Rightarrow \quad |B||B| = 1$$

$$\Rightarrow |B|^2 = 1 \quad \Rightarrow \quad |B| = \pm 1$$

The L.T. with $|B| = +1$ (or $|B| = -1$) is called proper (or improper) L.T. The set all L.Ts forms a group which is subgroup of a group called Poincare' group.

## Special Cases of L.Ts.

Consider some special cases of L.T.

(1) Space-time rotations.

The L.Ts given by $x'^i = A^i_j x^j$ represents space-time rotations.
The set of all such transformations form a subgroup of Poincare group. This s-group is called homogeneous L. group or simply L. group which is denoted by $L(6)$.

(2) Time Reversal transformations.

The linear Ts obtained by replacing $T$ by $-T$ (or $t$ by $-t$) such that

$$T' = -T, \quad x' = x, \quad y' = y, \quad z' = z \quad \text{——— (1)}$$

or $ct' = -ct, \; x' = x, \; y' = y, \; z' = z$

$\Rightarrow cdt' = -cdt, \; dx' = dx, \; dy' = dy, \; dz' = dz$

$\Rightarrow c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

$\Rightarrow ds'^2 = ds^2$

i.e. the linear transformation is a L.T.
It is called time reversal transformation and this transformation interchanges past and future.
It is to be noted that

$$|a_{ij}| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1$$

which shows that the time reversal trans. is an improper transformation and is not obtainable from the proper L.T.

Note that the product of two time reversel transformations

i.e. $AB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\delta^i_j)$ which is identity matri

Thus the time reversel transformations f
a subgroup of L. group.

③ Space Reflection.

The transformation given by

$x' = -x, \quad y' = -y, \quad z' = -z, \quad t' = t$

which changes the sign of every spa
coordinate and satisfies the equation

$c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2 dt^2 - (-dx) - d$

$= c^2 dt^2 - dx^2 - dy^2 - $

$\Rightarrow ds'^2 = ds^2$.

is called space reflection.

It is to be noted that

$|g_{ij}| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1$

which shows that the space reflection

is an improper transformation and is not obtainable from proper L.T.

Note that the product of two space reflection transformations

i.e. $AB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$= (\delta^i_j)$ which is identity matrix.

Thus the space reflection transformations form a subgroup of L. group.

(4) Space time translation

For $A^i_j = (\delta^i_j)$ the unit matrix of order $4 \times 4$, reduces the equation

$$x'^i = x^i + b^i \quad ; i = 0, 1, 2, 3.$$

gives us a set of transformations in 4-dimensional space time and this transf. changes $(x^i)$ - vector to a vector $(x'^i)$ which is translation of vector $(x^i)$ to $(x^i + b^i)$.

These transformations form a group of 4-vectors under addition. A/2016

Write a note on Lorentz Poincare' group

Poincare' group :- A/2013. S/2015

The entire group under which physical laws are invariant according to special Relativity is the ten parameter Poincare' group.

i.e. $P(10) = L(6) \otimes T(4)$

where $L(6)$ denotes the proper Lorentz group and the number 6 denotes the number of rotations, 3 for spatial rot and 3 for spatial temporal rotation. $T(4)$ denotes the group of translation of 4-vectors depending on four pa and $\otimes$ indicates that the element $L(6)$ do not commute with $T(4)$.
This may be seen by considering as

$$x'^i = A^i_j x^j \quad \text{———} (1)$$

and then

$$x''^i = x'^i + b^i \qquad \because \text{using } (1)$$

$$= A^i_j x^j + b^i$$

$$\Rightarrow x''^i = A^i_j x^j + b^i \quad \text{———} (2)$$

On the other hand if we define

$$\overset{*}{x}{}^i = x^i + b^i \quad \text{———} (3)$$

and then

$$\overset{**}{x}{}^i = A^i_j \overset{*}{x}{}^i \qquad \text{using } (3)$$

$$\Rightarrow x^{**i} = A^i_j (x^i + b^i)$$

$$\Rightarrow \overset{**}{x}{}^i = A^i_j x^i + A^i_j b^i \quad \text{———} (4)$$

From (2) and (4), we have

$$x''^i \neq \overset{**}{x}{}^i$$

The group of rotations in n-dimen

Euclidean space is called the orthogonal group and is denoted by $SO(n)$, where 'S' signifies that the matrix representing an element of this group has a unit determinant.

In Minkowski space the group is denoted by $SO(1,3)$, '1' referring to the time dimension and the '3' to the space dimensions.

Note: A/2012, Define an interval.

The interval $\Delta s$ in 4-dimensional space is defined by

$$\Delta s^2 = c^2(t_2-t_1)^2 - (x_2-x_1)^2 - (y_2-y_1)^2 - (z_2-z_1)^2$$

$$\Rightarrow \Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

$$= c^2\Delta t^2 - \Delta l^2$$

where $\Delta l$ is the spatial distance and $\Delta t$ is the time interval between the two events say $(ct_1, x_1, y_1, z_1)$ and $(ct_2, x_2, y_2, z_2)$.

The Null Cone Structure:–

Let us consider the infinitesimal vector $dx^i = (cdt, dx, dy, dz)$ which is invariant under Lorentz transformations, in the sense that its square magnitude remains invariant but not its components,

$$ds^2 = g_{ij}\, dx^i dx^j$$

$$= g'_{ij}\, dx'^i dx'^j$$

$$= ds'^2$$

A/2012, Classify the interval
Here, use $\Delta$ in the place of d (46)

A vector $dx^i$ may be classified into following three categories.

(i) Time like vector
(ii) Null / light like vector
(iii) Space like vector.

A/2018 Define (i), (ii), (

① Time like vector. A/2009

The 4-vector $dx^i$ is called time like $ds^2 > 0$.

$\Rightarrow c^2dt^2 - dl^2 > 0$

where $dl^2 = dx^2 + dy^2 + dz^2$.

$\Rightarrow c^2dt^2 > dl^2$

$\Rightarrow c^2 > \left|\frac{d\vec{l}}{dt}\right|^2$

or $\left|\frac{d\vec{l}}{dt}\right|^2 < c^2$.

$\Rightarrow$ the spatial distance between tw can be covered with velocity less than velocity of light.

Thus, for time like vectors the ma of velocity $\vec{v}$ is less than c. Th $dx^i$ can represent the actual path physical object in space over time a speed $\left|\frac{d\vec{l}}{dt}\right| < c$.

A/2009,

(ii) Null vector / Light like vector.

A 4-vector $dx^i$ is called null ve

like vector if $ds^2 = 0$

$$\Rightarrow c^2 dt^2 - dl^2 = 0$$

$$\Rightarrow c^2 dt^2 = dl^2$$

$$\Rightarrow c^2 = \left|\frac{d\vec{l}}{dt}\right|^2$$

$$\Rightarrow c^2 = \left|\frac{d\vec{l}}{dt}\right| \Rightarrow c = \left|\frac{d\vec{l}}{dt}\right|$$

For a null vector the magnitude of $\vec{v}$ is equal to $c$.

Thus $dx^i$ can represent the path of a physical object travelling at a light speed.

(iii) Space like vector. A/2009, A/2012

A 4-vector $dx^i$ is called space like vector if $ds^2 < 0$.

$$\Rightarrow c^2 dt^2 - dl^2 < 0$$

$$\Rightarrow c^2 dt^2 < dl^2$$

$$\Rightarrow c^2 < \left|\frac{d\vec{l}}{dt}\right|^2$$

$$\Rightarrow \left|\frac{d\vec{l}}{dt}\right| > c.$$

For a space like vector, the magnitude of $\vec{v}$ is greater than $c$, which is not possible for a physical object such as a particle.

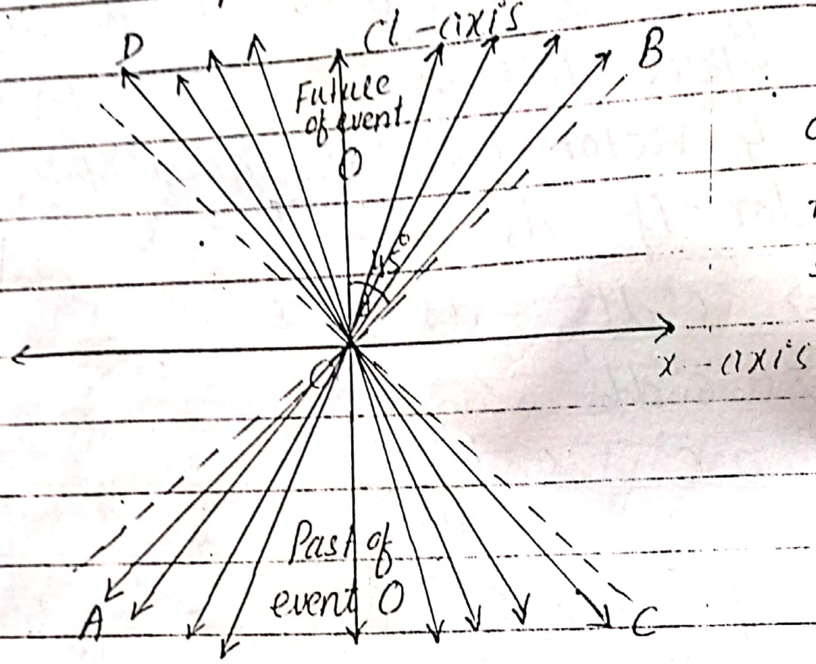
We can represent all the three types of vector together graphically. This graphical representation of 4-vector in space time

Note. Explain light cone structure.

(48)

S/2018. Write a note on null cone structure.

Continuum is called the null cone structure.
Since L.Ts leave $ds^2$ invariant, therefore,
we can always translate the origin in
space time to the origin of $dx^i$.
The set of events which are time like or
light like with respect to an event O a..
said to constitute the light cone of O.
To visulize this cone, consider two
dimensional space time diagram.




dotted represe.. the wo.. lines of space-.. event

Let us take the horizontal line as x-a..
and the vertical line as ct-axis. The
given event O is taken at the origin of
coordinate system. In particular, we sh..
assume O is a light pulse which expan..
in all directions immediately with the

speed c. Thus, in space-time diagram the distance covered by the light particle is $x = ct$

$$\Rightarrow \frac{x}{ct} = 1 = \tan 45^\circ$$

$\left\{ \begin{array}{l} y = mx \\ y/mx = 1 \end{array} \right.$

We draw this line and call it AOB which is the world line of a particle which moves with speed c and this world line makes an angle $\theta = 45^\circ$ with ct-axis.

The same is true about the world line COD. Thus any event on these lines AOB and COD is light like.

Since a material particle would always be moving with velocity $\vec{v} < c$, therefore, its world line will make an angle less than $45^\circ$ about ct-axis. So the motion of a material particle will always be represented by a line lying in the sectors AOC and BOD.

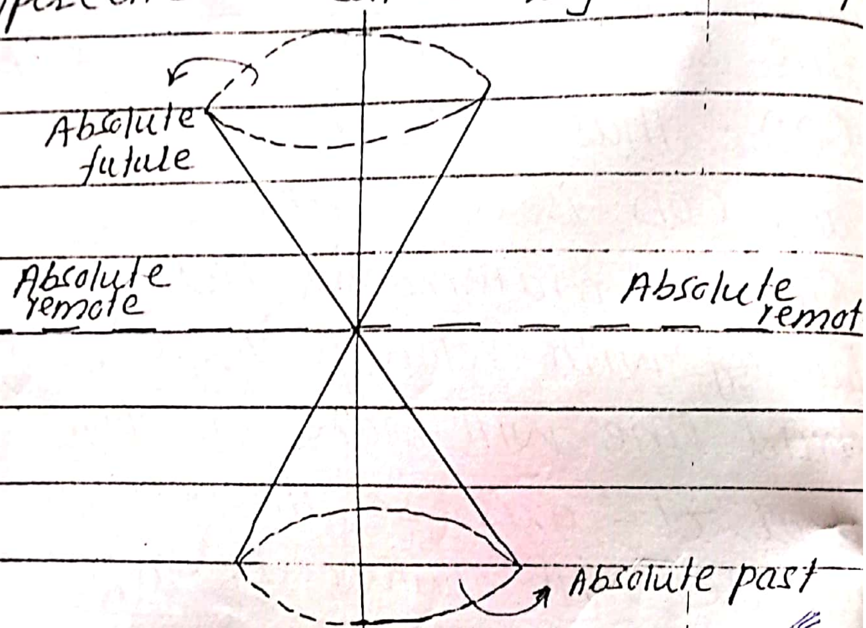
All the events in sector AOC will be in the past of event O and all the events in sector BOD will be in the future of event O.

The sectors BOC and AOD represent the world lines of the particles which moves with velocity $\vec{v} > c$. The events in these

regions BOC and AOD are said to be remote.

If we consider all the three space-coordinates $(x, y, z)$, then the three regi… of past, future and absolutely remote a… separated by the hypercone in the 4-dim… space. The axis of this hypercone coinci… with ct-axis.

This hypercone is called light cone / cone.




Expression for proper time as an integral.

Consider the invariant interval $ds^2$ in rest frame of the observer O.

Then $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ — (1)

For an observer in rest frame

$dx^i = (dx^1, dx^2, dx^3) = (0, 0, 0)$ ; t…

then (1) $\Rightarrow$ $ds^2 = c^2 d\tau^2$ — (2)

$$\Rightarrow d\tau^2 = ds^2/c^2$$

$\Rightarrow d\tau = ds/c$ is an invariant quantity.

Now an observer in $S'$ frame measures the time $dt'$ and the spatial displacement $d\vec{r}'$ and obtains the invariant quantity as

$$ds'^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$$

$$\Rightarrow ds'^2 = c^2 dt'^2 - (d\vec{r}')^2$$

But $ds'^2 = ds^2$, then

$$ds^2 = c^2 dt'^2 - (d\vec{r}')^2 \qquad (3)$$

Subsituting (2) in (3), we get

$$c^2 d\tau^2 = c^2 dt'^2 - (d\vec{r}')^2$$

$$\Rightarrow d\tau^2 = dt'^2 - \frac{1}{c^2}(u' dt')^2$$

$$= dt'^2 - \frac{1}{c^2} u'^2 dt'^2$$

$$= \left(1 - \frac{u'^2}{c^2}\right) dt'^2$$

$$\Rightarrow d\tau = \sqrt{1 - u'^2/c^2}\, dt'$$

Integrating, we get

$$\tau = \int \sqrt{1 - u'^2/c^2}\, dt'$$

which is the expression for proper time as measured by the moving observer.

**Examples:** A/2009, A/2014

The vectors $T, X, Y, Z$ are given by $T^a = (1,0,0,0)$, $X^a = (0,1,0,0)$; $Y^a = (0,0,1$ $Z = (0,0,0,1)$. Show that the only vanishin inner products between the vectors are

$$T^2 = -X^2 = -Y^2 = -Z^2 = 1.$$

Define $L^a = \frac{1}{\sqrt{2}}(T^a + Z^a)$, $N^a = \frac{1}{\sqrt{2}}(T^a - Z^a)$

$$M^a = \frac{1}{\sqrt{2}}(X^a + iY^a) \text{ and } \bar{M}^a = \frac{1}{\sqrt{2}}(X^a - iY$$

where $i = \sqrt{-1}$.

Treating $M^a$ and $\bar{M}^a$ as vectors, show that the four vectors are null and only non-vanishing inner products $L^a N_a = -M^a M_a =$

Solution.

As we know that

$$T^2 = g_{ab} T^a T^b = T^a (g_{ab} T^b) = T^a T_a$$

$$\Rightarrow T^2 = T^a T_a \quad \text{———} \quad (1)$$

Similarly

$$X^2 = X^a X_a \quad \text{———} \quad (2)$$

$$Y^2 = Y^a Y_a \quad \text{———} \quad (3)$$

$$Z^2 = Z^a Z_a \quad \text{———} \quad (4)$$

In Minkowski space

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$\Rightarrow g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1 \qquad g_{ab} = 0$$

$$(1) \Rightarrow T^2 = g_{ab} T^a T^b$$

$$\Rightarrow T^2 = g_{00}(T^0)^2 + g_{11}(T^1)^2 + g_{22}(T^2)^2 + g_{33}(T^3)$$

$= (T^0)^2 - (T^1)^2 - (T^2)^2 - (T^3)^2$

$= (1)^2 - (0)^2 - (0)^2 - (0)^2 = 1$

$\Rightarrow T^2 = 1 > 0.$

$\Rightarrow T = (T^a)$ is a time like vector.

Similarly;

$X^2 = g_{ab} X^a X^b$

$= g_{00} (X^0)^2 + g_{11} (X^1)^2 + g_{22} (X^2)^2 + g_{33} (X^3)^2$

$= 1 (0)^2 + (-1)(1)^2 - 1 (0)^2 - 1 (0)^2$

$\Rightarrow X^2 = -1 < 0$

$\Rightarrow X = (X^a)$ is a space like vector.

$Y^2 = g_{ab} Y^a Y^b$

$= g_{00} (Y^0)^2 + g_{11} (Y^1)^2 + g_{22} (Y^2)^2 + g_{33} (Y^3)^2$

$= 1 (0)^2 - 1 (0)^2 - (1)(1)^2 + (-1)(0)^2$

$= -1 < 0$

$\Rightarrow Y = (Y^a)$ is a space like vector.

and $Z^2 = g_{ab} Z^a Z^b$

$= g_{00} (Z^0)^2 + g_{11} (Z^1)^2 + g_{22} (Z^2)^2 + g_{33} (Z^3)^2$

$= 1 (0)^2 - 1 (0)^2 - (1)(0)^2 - 1 (1)^2$

$= -1 < 0$

$\Rightarrow Z = (Z^a)$ is a space like vector.

Hence $T^2 = -X^2 = -Y^2 = -Z^2 = 1.$ $X^a = (0,1,0,0)$

Now $M^a = \frac{1}{2}(X^a - iY^a)$ $iY^a = (0,0,1,0)$

$M^2 = g_{ab} M^a M^b$

$= g_{00} (M^0)^2 + g_{11} (M^1)^2 + g_{22} (M^2)^2 + g_{33} (M^3)^2$

$= (1)(0)^2 + (-1)\left(\frac{1}{\sqrt{2}}\right)^2 + (-1)\left(\frac{i}{\sqrt{2}}\right)^2 + (-1)(0)^2$

$$= -\frac{1}{2} - \left(-\frac{1}{2}\right) = -\cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} = 0$$

$\Rightarrow M^2 = 0$

$\Rightarrow M = (M^a)$ is a null vector.

$$N^2 = g_{ab} N^a N^b$$

$$= g_{00}(N^0)^2 + g_{11}(N^1)^2 + g_{22}(N^2)^2 + g_{33}(N^3)^2$$

$$= (1)\left(\frac{1}{\sqrt{2}}\right)^2 + (-1)(0)^2 + (-1)(0)^2 + (-1)\left(-\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

$\Rightarrow N = (N^a)$ is a null vector.

$$L^2 = g_{ab} L^a L^b$$

$$= g_{00}(L^0)^2 + g_{11}(L^1)^2 + g_{22}(L^2)^2 + g_{33}(L^3)^2$$

$$= (1)\left(\frac{1}{\sqrt{2}}\right)^2 + (-1)(0)^2 + (-1)(0)^2 + (-1)\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

$\Rightarrow L = (L^a)$ is a null-vector.

and $\bar{M}^2 = g_{ab} \bar{M}^a \bar{M}^b$

$$= g_{00}(\bar{M}^0)^2 + g_{11}(\bar{M}^1)^2 + g_{22}(\bar{M}^2)^2 + g_{33}(\bar{M}^3)^2$$

$$= (1)(0)^2 + (-1)\left(\frac{1}{\sqrt{2}}\right)^2 + (-1)\left(-\frac{i}{\sqrt{2}}\right)^2 + (-1)^2(0)$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

$\Rightarrow \bar{M} = (\bar{M}^a)$ is a null vector.

and ~~last~~ finally, we find

$$L^a N_a = -M^a \bar{M}_a = 1.$$

we know that

$$L^a N_a = g_{ab} L^a N^b$$

$\because$ using eq (1)

$$= g_{00}(L^0)(N^0) + g_{11}L^1N^1 + g_{22}L^2N^2 + g_{33}L^3N^3$$
$$= (1)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + (-1)(0)(0) - 1(0)(0) - 1\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)$$
$$= \frac{1}{2} - 0 - 0 + \frac{1}{2} = 1$$

$$\Rightarrow L^a N_a = 1.$$

and $M^a \bar{M}_a = g_{ab} M^a \bar{M}^b$

$$= g_{00} M^0 \bar{M}^0 + g_{11} M^1 \bar{M}^1 + g_{22} M^2 \bar{M}^2 + g_{33} M^3 \bar{M}^3$$
$$= (1)(0)(0) + (-1)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + (-1)\left(\frac{i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right) - 1(0)(0)$$
$$= \frac{-1}{2} - \frac{1}{2} = -1$$

$$\Rightarrow M^a \bar{M}_a = -1 \quad \text{or} \quad -M^a \bar{M}_a = 1$$

Hence $L^a N_a = -M^a \bar{M}_a = 1.$

**Example 2.** A/2018 $\Rightarrow X^\alpha = (0,0,1,0)$, $Y^\alpha = (2,1,0,1)$, $Z^\alpha = (1,1,0,1$

Classify the following vectors as time-like, null or space-like.

$A^a = (-1, 4, 0, 1)$, $B^a = (2, 0, -1, 1)$

and $C^a = (2, 0, -2, 0)$.

Solution.

Given that

$$A^a = (A^0, A^1, A^2, A^3) = (-1, 4, 0, 1)$$
$$B^a = (B^0, B^1, B^2, B^3) = (2, 0, -1, 1)$$
$$C^a = (C^0, C^1, C^2, C^3) = (2, 0, -2, 0)$$

Therefore,

$$A^2 = g_{ab} A^a A^b$$
$$= g_{00}(A^0)^2 + g_{11}(A^1)^2 + g_{22}(A^2)^2 + g_{33}(A^3)^2$$

In Minkowski space

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

$$\Rightarrow g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1$$

Thus $A^2 = (1)(-1)^2 + (-1)(4)^2 + (-1)(0) + (-1)($

$$= 1 - 16 - 1 \quad = -16 < 0$$

$\Rightarrow A = (A^a)$ is a space like vector.

$$B^2 = g_{00}(B^0)^2 + g_{11}(B^1)^2 + g_{22}(B^2)^2 + g_{33}(B$$

$$= 1(2)^2 - 1(0)^2 - 1(-1)^2 - 1(1)^2$$

$$= 4 - 1 - 1 \quad = 2 > 0$$

$\Rightarrow B = (B^a)$ is a time like vector.

and $C^2 = g_{00}(C^0)^2 + g_{11}(C^1)^2 + g_{22}(C^2)^2 + g_{33}(C^3)^2$

$$= 1(2)^2 - 1(0)^2 - 1(-2)^2 - 1(0)^2$$

$$= 4 - 4 \quad = 0$$

$\Rightarrow C = (C^a)$ is a null vector.

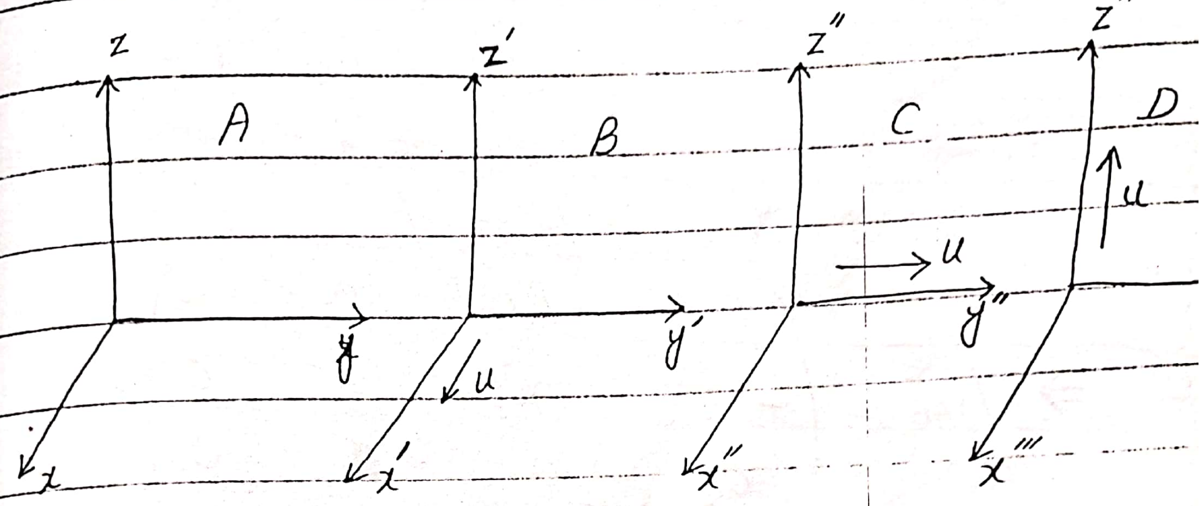
**Problem.**

Let an observer A see observer B movi with a speed $u$ in the $x$-direction, obs B see observer C moving in the $y$-direc and observer C see observer D moving the $z$-direction. Work out the L.Ts m of the motion of D relative to A ar of A relative to D. Are the two ma inverses of each other?

Solution.

Let us assume that an observer A se

observer B moving with a speed $u$ in the $x$-direction, observer B see observer C moving in the $y$-direction and observer C see observer D moving in the $z$-direction as shown in the following Fig.




The general L.Ts matrix is given by

$$\Lambda = \begin{pmatrix} \gamma & -\beta_1\gamma & -\beta_2\gamma & -\beta_3\gamma \\ -\beta_1\gamma & 1+\beta_1^2(\gamma-1)/\beta^2 & \beta_1\beta_2(\gamma-1)/\beta^2 & \beta_1\beta_3(\gamma-1)/\beta^2 \\ -\beta_2\gamma & \beta_1\beta_2(\gamma-1)/\beta^2 & 1+\beta_2^2(\gamma-1)/\beta^2 & \beta_2\beta_3(\gamma-1)/\beta^2 \\ -\beta_3\gamma & \beta_1\beta_3(\gamma-1)/\beta^2 & \beta_2\beta_3(\gamma-1)/\beta^2 & 1+\beta_3^2(\gamma-1)/\beta^2 \end{pmatrix}$$

For the motion of observer B relative to A, the uniform velocity of B w.r.t. A is

$\vec{u} = (u, 0, 0)$ (motion along $x$-axis)

$\Rightarrow \vec{\beta} = \dfrac{\vec{u}}{c} = (u/c, 0, 0) = (\beta_1, \beta_2, \beta_3)$

and

$\gamma = (1 - u^2/c^2)^{-1/2} = (1-\beta^2)^{-1/2}$.

Then the L.Ts matrix of B w.r.t. A is

$$\Lambda_{AB} = \begin{pmatrix} \gamma & -\beta_1\gamma & 0 & 0 \\ -\beta_1\gamma & 1+\beta_1^2(\gamma-1)/\beta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $1+\beta_1^2(\gamma-1)/\beta^2 = 1 + \frac{u^2}{c^2}(\gamma-1)/u^2/c^2$

$= 1 + (\gamma - 1)$

$= \cancel{1} + \gamma - \cancel{1}$

$\Rightarrow 1+\beta_1^2(\gamma-1)/\beta^2 = \gamma$

$$\Rightarrow \Lambda_{AB} = \begin{pmatrix} \gamma & -\beta_1\gamma & 0 & 0 \\ -\beta_1\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For ~~relat~~ the motion of observer C rel to B, the uniform velocity of C w.r is $\vec{u} = (0, u, 0)$ (motion along y-ax

$\Rightarrow \vec{\beta} = \vec{u}/c = (0, u/c, 0) = (\beta_1, \beta_2, \beta_3)$

$\gamma = (1-\beta^2)^{-1/2}$

Then the L.Ts matrix of C w.r.t. B

$$\Lambda_{BC} = \begin{pmatrix} \gamma & 0 & -\beta_2\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Similarly, for the motion of observer relative to C, the uniform velocity of D

$C$ is $\vec{u} = (0, 0, u)$ (motion along z-axis)

$\Rightarrow \vec{\beta} = \vec{u}/c = (0, 0, u/c) = (\beta_1, \beta_2, \beta_3)$

Then the L.Ts matrix of D w.r.t. C is

$$\Lambda_{CD} = \begin{pmatrix} \gamma & 0 & 0 & -\beta_3\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma & 0 & 0 & 1 \end{pmatrix}$$

## Position 4-vector.

Let us consider a particle moving with velocity $\vec{V} = (V_x, V_y, V_z)$ in the S-frame. The position of this particle at any instant $t$ from $\underline{O}$ is represented by

$$x^i = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

where $x^0 = ct$ and $\underline{O} = (0,0,0,0)$.

The vector $(x^i)$ is called position 4-vector of the particle in 4-dimensional space.

## 4-Velocity :- A/2009, A/2010, S/2018

The rate of change of 4-position vector w.r.t. the proper time is called 4-velocity of the particle.

i.e. $$\dot{x}^i = v^i = \frac{dx^i}{d\tau} = \frac{d}{d\tau}(ct, x, y, z)$$

$$\Rightarrow \dot{x}^i = \left(\frac{c\,dt}{d\tau}, \frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}\right) \quad \text{--- } (1)$$

where $\tau$ is proper time.

But $$ds^2 = ds'^2$$

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$$

$$\Rightarrow c^2dt^2 - d\vec{x}\cdot d\vec{x} = c^2d\tau^2 - 0 - 0 - 0$$

($\because dx' = dy' = dz' = 0$ in the frame in which proper time is being observed)

$$\Rightarrow c^2dt^2\left(1 - \frac{|dx/dt|^2}{c^2}\right) = c^2d\tau^2$$

$$\frac{dt^2}{d\tau^2} = (1 - v^2/c^2)^{-1} = \gamma^2 \qquad (2)$$

$$\Rightarrow \frac{dt}{d\tau} = \gamma.$$

Note that

$$\left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\frac{dt}{d\tau}$$

$$= (V_x, V_y, V_z)\,\gamma$$

$$= \gamma\vec{V} \qquad (3)$$

Subsituting (2) and (3) in (1), we get

$$\dot{x}^i = (\gamma c, \gamma\vec{V}).$$

The squared magnitude of four-velocity is given by

$$V^2 = g_{ij} V^i V^j$$

$$= g_{00}(V^0)^2 + g_{11}(V^1)^2 + g_{22}(V^2)^2 + g_{33}$$

In Minkowski space, we have

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1$$

Thus

$$V^2 = (1)(\gamma c)^2 + (-1)(\gamma V_x)^2 + (-1)(\gamma V_y)^2 + (-1)($$

$$= \gamma^2 c^2 - \gamma^2(V_x^2 + V_y^2 + V_z^2)$$

$$= \gamma^2 c^2 - \gamma^2 v^2$$

$$= \gamma^2 c^2 (1 - v^2/c^2) = \gamma^2 c^2 \gamma^{-2}$$

$$\Rightarrow V^2 = c^2$$

The 4-velocity vector $V^i = (V^0; V^1, V^2, V$ transforms in the same way as th

position 4-vector $x^i = (x^0, x^1, x^2, x^3)$.

i.e. $t' = \gamma(t - vx/c^2)$

$\Rightarrow x^{0'} = \gamma(x^0 - \frac{v x^1}{c})$ $\quad \because x^0 = ct$.

$\Rightarrow V^{0'} = \gamma(V^0 - \frac{v V^1}{c})$

and $x' = \gamma(x - vt) = \gamma(x - \frac{vct}{c})$

$\Rightarrow x^{1'} = \gamma(x^1 - \frac{v}{c} x^0)$

$\Rightarrow V^{1'} = \gamma(V^1 - \frac{v}{c} V^0)$

$y' = y \quad ; \quad z' = z$

$V^{2'} = V^2 \quad ; \quad V^{3'} = V^3$

**4-momentum:-** A/2009, A/2010

Let us consider a particle in motion. Let $V^i$ be its 4-velocity. Then the product of the velocity 4-vector with a 4-scalar $m_0$ is again a 4-vector and is called 4-momentum denoted by $p^i$ as

$$p^i = m_0 V^i = m_0(V^0, V^1, V^2, V^3)$$
$$= m_0(\gamma c, \gamma \vec{v})$$
$$= m_0 \gamma (c, v_x, v_y, v_z)$$

Let us introduce $m = m_0\gamma = \frac{m_0}{\sqrt{1 - v^2/c^2}}$ which is called the relativistic mass, dependent on the velocity of the moving object,

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$m_0$ is called rest mass, showing that m is not a constant quantity but rath[er] depends on the velocity of particle.

So

$$p^i = \frac{m_0}{\sqrt{1-v^2/c^2}}(c, v_x, v_y, v_z)$$

$$\Rightarrow p^0 = \frac{m_0}{\sqrt{1-v^2/c^2}}\, c = mc$$

$$p^1 = \frac{m_0}{\sqrt{1-v^2/c^2}}\, v_x = m v_x$$

$$p^2 = \frac{m_0}{\sqrt{1-v^2/c^2}}\, v_y = m v_y$$

$$p^3 = \frac{m_0}{\sqrt{1-v^2/c^2}}\, v_z = m v_z$$

$\Rightarrow$ $p^0$ gives us temporal component of [4-] momentum $p^i$ and $p^1, p^2, p^3$ are spa[tial] components of 4-momentum $p^i$.

① It is to be noted that

$$p^0 = \frac{m_0 c}{\sqrt{1-v^2/c^2}} = mc = \frac{mc^2}{c} = E/c$$

$$\Rightarrow p^0 = E/c$$

$$\Rightarrow E = mc^2$$

gives us the famous mass-energy rel[ation]

and $p^0 = \frac{m_0 c}{\sqrt{1 - V^2/c^2}} = \frac{m_0 c^2}{c\sqrt{1 - V^2/c^2}} = \frac{E_0}{c\sqrt{1 - V^2/c^2}}$

$\Rightarrow E_0 = m_0 c^2$ is called rest mass energy or residual energy.

(2) Also note that the K.E. of the particle in this case given by

$$T = E - E_0$$
$$= mc^2 - m_0 c^2$$
$$= m_0 \gamma c^2 - m_0 c^2 \qquad \because \gamma = \sqrt{1 - V^2/c^2}$$
$$= m_0 c^2 (\gamma - 1)$$
$$= m_0 c^2 \left[ \left(1 - \frac{V^2}{c^2}\right)^{-1/2} - 1 \right]$$

$$= m_0 c^2 \left[ \left(1 + \frac{1}{2} V^2/c^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} V^4/c^4 - \frac{(-\frac{1}{2})(-3/2)(-5/2)}{3!} V^6/c^6 + \cdots \right) - 1 \right]$$ By Binomial expansion

$$= m_0 c^2 \left( V^2/2c^2 + \frac{3}{8} V^4/c^4 + \frac{5}{16} V^6/c^6 + \cdots \right)$$

$$= \frac{1}{2} m_0 V^2 + \frac{3}{8} m_0 \frac{V^4}{c^2} + \frac{5}{16} m_0 \frac{V^6}{c^4} + \cdots$$

$$= \frac{1}{2} m_0 V^2 \left( 1 + \frac{3}{4} V^2/c^2 + \frac{5}{8} V^4/c^4 + \cdots \right)$$

For $V << c$, $V/c \to 0$, $V^2/c^2 \to 0$, $V^4/c^4 \to 0$ and so on.

Hence, $T = \frac{1}{2} m_0 v^2$ represents classical Newtonian limit of relativistic result.

The relativistic correction to classical expression for K.E. is given by

$$\gamma - 1 = (1 - v^2/c^2)^{-1/2} - 1$$

$$= 1 + \frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + O\left(\frac{v}{c}\right)^6 - 1$$

$$= \frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + O\left(\frac{v}{c}\right)^6$$

which yields

$$T = (\gamma - 1) E_0$$

$$= (\gamma - 1) m_0 c^2$$

$$= \left[\frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + O\left(\frac{v}{c}\right)^6\right] m_0 c^2$$

$$= \frac{1}{2} m_0 v^2 + \frac{3}{8} m_0 \frac{v^4}{c^2} + O\left(v/c\right)^4$$

$$\simeq \frac{1}{2} m_0 v^2 \left(1 + \frac{3}{4} v^2/c^2 + O\left(v/c\right)^4\right)$$

If we take $m$ instead of $m_0$, then

$$T = (\gamma - 1) E_0$$

$$= (\gamma - 1) m_0 c^2$$

$$= \frac{(\gamma - 1)}{\gamma} m_0 \gamma c^2$$

$$= (1 - 1/\gamma)\, m c^2 \qquad \because m = m_0 \gamma$$

$$\Rightarrow T = (1-\gamma^{-1})mc^2$$

$$= \left[1-(1-v^2/c^2)^{+1/2}\right]mc^2$$

$$= \left[1-(1-\tfrac{1}{2}v^2/c^2 - \tfrac{1}{8}\tfrac{v^4}{c^4} - O(v/c)^6\right]mc^2$$

$$= \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{1}{8}\frac{v^4}{c^4} + O(v/c)^6\right)mc^2$$

$$= \frac{1}{2}mv^2\left(1+\frac{1}{4}v^2/c^2 + O(v/c)^4\right)$$

where $m = m_0\gamma$ is relativistic mass.

(3) The squared magnitude of 4-momentum in Minkowski space is

$$p^2 = g_{ij}\,p^i p^j$$

$$= g_{00}(p^0)^2 + g_{11}(p^1)^2 + g_{22}(p^2)^2 + g_{33}(p^3)^2$$

$$= (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$$

$$= (mc)^2 - (mv_x)^2 - (mv_y)^2 - (mv_z)^2$$

$$= m^2c^2 - m^2(v_x^2 + v_y^2 + v_z^2)$$

$$= m^2c^2 - m^2v^2$$

$$= m^2c^2(1-v^2/c^2)$$

$$= m^2c^2\gamma^{-2}$$

$$= m_0^2\gamma^2c^2\gamma^{-2} \qquad \because m = m_0\gamma.$$

$$= m_0^2c^2.$$

A/2013, A/20

(4) Energy Momentum relation. A/2009, A/2010

The energy momentum relation can be observed from squared magnitude of 4-momentum in Minkowski space.

$$p^2 = g_{ij}\, p^i p^j$$
$$= (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$$
$$\Rightarrow m_0^2 c^2 = (p^0)^2 - \vec{p} \cdot \vec{p} \qquad \because p^0 = E/c$$
$$= \left(\frac{E}{c}\right)^2 - p^2$$
$$= E^2/c^2 - p^2$$
$$= \frac{E^2 - p^2c^2}{c^2}$$
$$\Rightarrow m_0^2 c^4 = E^2 - p^2 c^2$$
$$\Rightarrow E^2 = m_0^2 c^4 + p^2 c^2 .$$

which is the desired energy momen relation.

(5) L.Ts for 4-momentum Components

Let $p^i$ and $p'^i$ be the 4-momentum in the two Coordinate system, then th L.Ts Connecting $p^i$ and $p'^i$ is given b

$$p'^0 = \gamma\left(p^0 - p^1 \frac{v}{c}\right)$$

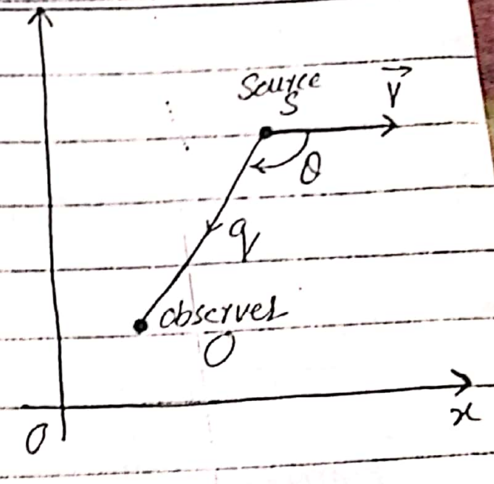
$$p'^1 = \gamma\left(p^1 - \frac{v}{c} p^0\right)$$
$$p'^2 = p^2 \quad ; \quad p'^3 = p^3 .$$

Dopples Shift in Relativity. 5/2018.

The Change in frequency due to th relative motion between the source and observes is called the relati

**Dopples shift :—**

Explanation.




Let us consider an observer O observing the light emitted by source S which is itself moving with velocity $\vec{V}$ and emitting light of frequency $\nu'$.

Further assume that the motion of the source S is taken along x-axis and making angle θ with the line of sight of the observer.

The energy momentum relation is given by

$$E^2 = m_0^2 c^4 + p^2 c^2 \quad \text{——— (I)}$$

where $m_0$ is rest mass of the photon and p is its momentum.

But the rest mass of the photon is zero, therefore,

$$(1) \Rightarrow E^2 = 0 + p^2c^2$$

$$\Rightarrow E^2 = p^2c^2$$

$$\Rightarrow E = pc \quad \text{——— (2)}$$

$$\Rightarrow p = E/c$$

Since we know that, for photon.

$$E = h\nu \quad \text{———} (3)$$

where $h$ is Plank's Constant and $\nu$ the frequency of photon observed by observer O.

$$(3) \text{ in } (2) \Rightarrow \; p = \frac{h\nu}{c} \quad \text{———} (4)$$

We further assume that $q^i$ is 4-momentum in the observer frame, then

$$q^i = (q^0, q^1, q^2, q^3)$$
$$= (q, q\cos\theta, q\sin\theta, \overset{\text{zero}}{\vec{0}})$$

where $q$ is the momentum of photon in the observer frame making an angle $\theta$ with the line of sight.

Transforming the momentum Components using Lorentz transformation as for the position 4-vector, we have

$$q'^0 = \gamma\left(q^0 - q^1\frac{v}{c}\right)$$

$$\Rightarrow q' = \gamma\left(q - q\cos\theta\,\frac{v}{c}\right)$$

$$= \gamma q\left(1 - \frac{v}{c}\cos\theta\right)$$

$$\Rightarrow \frac{h\nu'}{c} = \gamma\left(1 - \frac{v}{c}\cos\theta\right)\left(\frac{h\nu}{c}\right) \quad \because \text{using } (4)$$

$$\Rightarrow \nu' = \gamma\left(1 - \frac{v}{c}\cos\theta\right)\nu$$

$\Rightarrow \lambda > \lambda'$

$\Rightarrow \lambda - \lambda' > 0$

Thus, the wavelength is increased.
Since blue light has a shorter wavelength and red light has a longer wavelength, therefore, the shift is towards the red end of the spectrum. It is called a red shift.

(2) Observer and the source moving together.
This is possible when $\theta = 0$.

$$(5) \Rightarrow \frac{\nu'}{\nu} = \gamma\left(1 - \frac{v}{c}\cos 0^\circ\right)$$

$$= \gamma(1 - v/c) = \frac{1 - v/c}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{1 - v/c}{\sqrt{1 - v/c}\sqrt{1 + v/c}}$$

$$= \sqrt{\frac{1 - v/c}{1 + v/c}} = \sqrt{\frac{c - v}{c + v}} < 1$$

$\because c - v < c + v$.

which gives blue shif.

(3) Transverse Doppler effect.
When the source is moving at an an $\theta = \pi/2$ to the line of sight of the observer.

$$\Rightarrow \frac{\nu'}{\nu} = \gamma\left(1 - \frac{v}{c}\cos\theta\right)$$

= ratio of emitted to observed frequency.

Special Cases.

(1) Observer and Source are moving apart.

This is possible when $\theta = \pi$

$$(5) \Rightarrow \frac{\nu'}{\nu} = \gamma\left(1 - \frac{v}{c}\cos\pi\right)$$

$$= \gamma(1 + v/c) = \frac{1 + v/c}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{1 + v/c}{\sqrt{1 - v/c}\sqrt{1 + v/c}}$$

$$= \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}} = \sqrt{\frac{1 + v/c}{1 - v/c}} = \sqrt{\frac{c+v}{c-v}}$$

$\because c+v > c-$

$$\Rightarrow \frac{\nu'}{\nu} > 1 \Rightarrow \nu' > \nu$$

i.e. if the motion is radially away f the observer, the frequency is decrea

Now, we know that the wavelength given by $\lambda = c/\nu$

$$\nu' > \nu \Rightarrow \frac{1}{\nu} > \frac{1}{\nu'} \Rightarrow \frac{c}{\nu} > \frac{c}{\nu'}$$

Then (5) $\Rightarrow \frac{\nu'}{\nu} = \gamma\left(1 - \frac{v}{c}\cos\frac{\pi}{2}\right)$

$$= \gamma(1-0)$$

$$\Rightarrow \frac{\nu'}{\nu} = \frac{1}{\sqrt{1-v^2/c^2}} > 1.$$

which is due to time dilation.

(4) Condition for no Doppler shift.

Let us find $\theta$ for which there is no Doppler shift.

i.e. $\nu' = \nu$.

$$\Rightarrow \frac{\nu'}{\nu} = 1 \quad \text{---} \quad (*)$$

Subsituting (5) in (*), we get

$$\gamma\left(1 - \frac{v}{c}\cos\theta\right) = 1$$

$$\Rightarrow 1 - \frac{v}{c}\cos\theta = \sqrt{1 - v^2/c^2}$$

On Squaring, we get

$$\left(1 - v/c\cos\theta\right)^2 = 1 - v^2/c^2$$

$$\Rightarrow 1 + \frac{v^2}{c^2}\cos^2\theta - 2\frac{v}{c}\cos\theta = 1 - v^2/c^2$$

$$\Rightarrow \cancel{1} + \frac{v^2}{c^2}\cos^2\theta - 2\frac{v}{c}\cos\theta - \cancel{1} + v^2/c^2 = 0$$

$$\Rightarrow \frac{v^2}{c^2}\cos^2\theta - 2\frac{v}{c}\cos\theta + \frac{v^2}{c^2} = 0$$

which is quadratic equation in $\cos\theta$.

By quadratic formula, we have

$$\cos\theta = \frac{2v/c \pm \sqrt{4v^2/c^2 - 4(v^2/c^2)(v^2/c^2)}}{2(v^2/c^2)}$$

$$= \frac{2v/c \pm 2v/c\sqrt{1 - v^2/c^2}}{2(v^2/c^2)}$$

$$= \frac{1 \pm \sqrt{1 - v^2/c^2}}{v/c}$$

$$= \frac{c}{v}\left[1 \pm \sqrt{1 - v^2/c^2}\right]$$

$$= c/v \pm c/v\sqrt{1 - v^2/c^2}$$

$$= c/v \pm \sqrt{c^2/v^2\left(1 - v^2/c^2\right)}$$

$$= c/v \pm \sqrt{c^2/v^2 - 1}$$

But $|\cos\theta| \leq 1$.

Cos θ re below +1

$$\Rightarrow \cos\theta = \frac{c}{v} - \sqrt{\frac{c^2}{v^2} - 1}$$

$(\because$ discard $c/v + \sqrt{c^2/v^2 - 1} > 1.)$

$$\Rightarrow \cos\theta = \frac{c}{v} - \left(\frac{c^2}{v^2} - 1\right)^{1/2}$$

$$= \frac{c}{v} - \frac{c}{v}\left(1 - v^2/c^2\right)^{1/2}$$

$$= \frac{c}{v}\left[1 - \left(1 - v^2/c^2\right)^{1/2}\right]$$

$$= c/v\left[1 - \left\{1 + \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{1/2(1/2 - 1)}{2!}\left(-v^2/c^2\right.\right.\right.$$

$$= \frac{c}{v}\left[\cancel{1} - \cancel{1} + \frac{1}{2} v^2/c^2 + \frac{1}{8} v^4/c^4 + \cdots\right]$$

$$= \frac{c}{v}\left[\frac{1}{2} v^2/c^2 + \frac{1}{8} v^4/c^4 + \cdots\right]$$

$$= \frac{c}{v}\left(\frac{v^2}{2c^2}\right) \quad \text{neglecting higher powers}$$

$$= v/2c$$

$$\Rightarrow \theta = \cos^{-1}(v/2c)$$

Example.

A distant galaxy in the constellation Hydra تقریباً is receding from the earth at speed $6.12 \times 10^7$ m/s. By how much is a green spectral line of wavelength $\lambda_0 = 500$nm emitted by this galaxy shifted towards the red end.

Solution.

Given that $V = 6.12 \times 10^7$ m/s

and $\lambda_0 = 500 \times 10^{-9}$ m

So by the formula for Doppler shift in light given by

$$\frac{\nu'}{\nu} = \sqrt{\frac{c+v}{c-v}}$$

$$\Rightarrow \frac{\nu_0}{\nu} = \sqrt{\frac{c+v}{c-v}}$$

$$\Rightarrow \frac{c/\lambda_0}{c/\lambda} = \sqrt{\frac{c+v}{c-v}}$$

$$\Rightarrow \frac{\lambda}{\lambda_0} = \sqrt{\frac{c+v}{c-v}}$$

$$\Rightarrow \lambda = \lambda_0 \sqrt{\frac{c+v}{c-v}}$$

$$= (500 \times 10^{-9}) \sqrt{\frac{3\times10^8 + 6.12\times10^7}{3\times10^8 - 6.12\times10^7}}$$

$$= (500 \times 10^{-9}) \sqrt{\frac{30\times10^7 + 6.12\times10^7}{30\times10^7 - 6.12\times10^7}}$$

$$= (500 \times 10^{-9}) \sqrt{\frac{36.12 \times10^7}{23.88\times10^7}}$$

$$= 500 \times 10^{-9} \sqrt{\frac{36.12}{23.88}}$$

$$= 500 \times 10^{-9} \sqrt{1.513}$$

$$= 500 \times 10^{-9} \times 1.229$$

$$= 614.93 \times 10^{-9}$$

$\Rightarrow \lambda = 614.93 \text{ nm} \approx 615\text{nm}$

which is orange part of spectrum.

$$\Delta\lambda = \lambda - \lambda_0 = (615 - 500)\times10^{-9}$$

$$= 115 \text{ nm}$$

This galaxy is believed to be 3.6 [illegible]
light years away.

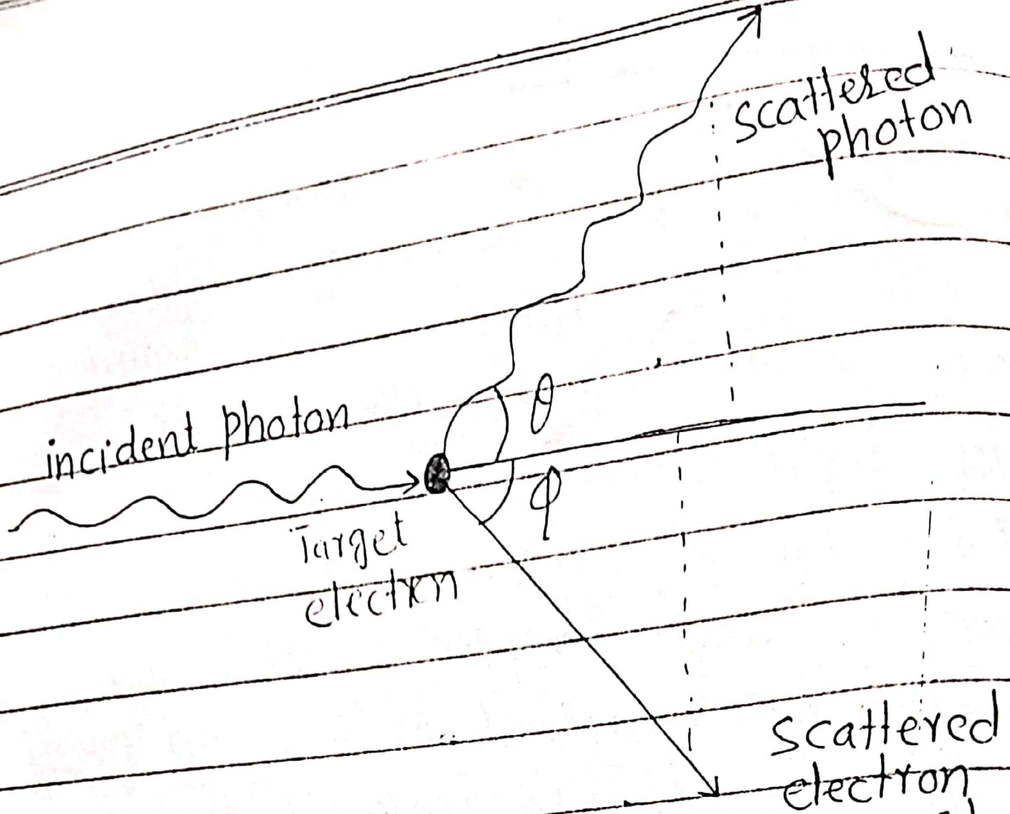
## The Compton Effect:-

According to the quantum theory of light, photons behave like particles except for their lack of rest mass. We shall examine the collision of photons with electrons.

Consider light (photons) incident on electrons at rest. The scattering of a photon by an electron is called Compton effect. Energy and momentum are conserved in such an event and as a result the scattered photon has less energy than the incident photon.

We want to study this effect in relativistic term.

For this purpose, we consider a collision between a single photon and an electron. Let us assume that $p_1^i$ and $q_1^i$ represent the momentum 4-vectors of an electron and that of photon before collision and furthermore $p_2^i$ and $q_2^i$ represent the momentum 4-vector of electron and photon after collision. Let us assume that after the collision, the photon goes off at an angle $\theta$ relative to the direction of motion while the electron goes off at an angle $\varphi$, measured in opposite sense to $\theta$.




Thus, before the collision, momentum of an electron is

$$p_1^i = (p_1^0, p_1^1, p_1^2, p_1^3) = (E/c, m_e \vec{V})$$

$$= \left(\frac{m_e c^2}{c}, m_e \vec{V}\right)$$

$$= (m_e c, 0, 0, 0) \quad \because \vec{V} = 0 \text{ for the at res}$$

$$\Rightarrow p_1^i = (m_e c, 0, 0, 0) \longrightarrow (1)$$

and momentum of photon is

$$q_1^i = (q_1^0, q_1^1, q_1^2, q_1^3)$$

$$= (E/c, q_x, q_y, q_z)$$

$$= (E/c, q_x, 0, 0) \quad \because \text{motion } \cancel{x\text{-axis}}$$

$$= \left(\frac{m_p c^2}{c}, m_p V_x, 0, 0\right)$$

$$= (m_p c, m_p V_x, 0, 0)$$

$$\Rightarrow q_1^i = (q_1, q_1, 0, 0) \longrightarrow (2)$$

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After the collision, the momentum of electron is

$$p_2^i = (p_2^0, p_2^1, p_2^2, p_2^3)$$
$$= (E/c, p\cos\varphi, -p\sin\varphi, 0) \longrightarrow (3)$$

and the momentum of photon is

$$q_2^i = (q_2^0, q_2^1, q_2^2, q_2^3)$$
$$= (q', q'\cos\theta, q'\sin\theta, 0) \longrightarrow (4)$$

By the law of conservation of momentum

$$p_1^i + q_1^i = p_2^i + q_2^i \longrightarrow (5)$$

$$\Rightarrow p_2^i = p_1^i + q_1^i - q_2^i \longrightarrow (6)$$

and

$$p_2^j = p_1^j + q_1^j - q_2^j \longrightarrow (7)$$

Thus, in Minkowski Space, we have

$$g_{ij} p_2^i p_2^j = g_{ij} (p_1^i + q_1^i - q_2^i)(p_1^j + q_1^j - q_2^j)$$

$$= g_{ij} p_1^i p_1^j + g_{ij} p_1^i (q_1^j - q_2^j)$$
$$+ g_{ij} p_1^j (q_1^i - q_2^i) + g_{ij} (q_1^i - q_2^i)(q_1^j - q_2^j)$$

$$= g_{ij} p_1^i p_1^j + g_{ij} p_1^i (q_1^j - q_2^j)$$

$i, j$ dummy index

$$+ g_{ij} p_1^i (q_1^j - q_2^j) + g_{ij} (q_1^i - q_2^i)(q_1^j - q_2^j)$$

$$\Rightarrow g_{ij} p_2^i p_2^j = g_{ij} p_1^i p_1^j + 2 g_{ij} p_1^i (q_1^j - q_2^j)$$
$$+ g_{ij} (q_1^i - q_2^i)(q_1^j - q_2^j) \longrightarrow (8)$$

But we know that the squared magnitude

square of

of momentum 4-vector is the (rest) ma[ss] multiplied by $c^2$.

Thus, $g_{ij} p_2^i p_2^j = m_e c^2$

and $g_{ij} p_1^i p_1^j = m_e c^2$

$$⑧ \Rightarrow \cancel{m_e c^2} = \cancel{m_e c^2} + 2 g_{ij} p_1^i (q_1^j - q_2^j)$$

$$+ g_{ij} (q_1^i - q_2^i)(q_1^j - q_2^j)$$

$$\Rightarrow 2 g_{ij} p_1^i (q_1^j - q_2^j) + g_{ij} (q_1^i - q_2^i)(q_1^j - q_2^j)$$

$$\Rightarrow 2 g_{00} p_1^0 (q_1^0 - q_2^0) + 2 g_{11} p_1^1 (q_1^1 - q_2^1)$$

$$+ 2 g_{22} p_1^2 (q_1^2 - q_2^2) + 2 g_{33} p_1^3 (q_1^3 - q_2^3)$$

$$+ g_{00} (q_1^0 - q_2^0)(q_1^0 - q_2^0) + g_{11} (q_1^1 - q_2^1)(q_1^1 - q$$

$$+ g_{22} (q_1^2 - q_2^2)(q_1^2 - q_2^2) + g_{33} (q_1^3 - q_2^3)(q_1^3 -$$

Putting $g_{00} = 1$; $g_{11} = g_{22} = g_{33} = -1$, we g[et]

$$2 p_1^0 (q_1^0 - q_2^0) - 2 p_1^1 (q_1^1 - q_2^1) - 2 p_1^2 (q_1^2 -$$

$$-2 p_1^3 (q_1^3 - q_2^3) + (q_1^0 - q_2^0)^2 - (q_1^1 - q_2^1)^2$$

$$-(q_1^2 - q_2^2)^2 - (q_1^3 - q_2^3)^2 = 0 \longrightarrow ($$

Subsituting eqs ①, ② and ④ in eq ⑨, we get

$$2 m_e c (q - q') - 0 - 0 - 0 + (q - q')^2$$

$$-(q - q' \cos\theta)^2 - (0 - q' \sin\theta)^2 - (0 - 0)^2 = 0$$

$$\Rightarrow 2 m_e c (q - q') + (q - q')^2 - (q - q' \cos\theta)^2 - (-$$

$$\Rightarrow 2m_e c(q-q') + q^2 + q'^2 - 2qq' - (q^2 + q'^2\cos^2\theta - 2qq'\cos\theta) - q'^2\sin^2\theta = 0$$

$$\Rightarrow 2m_e c(q-q') + \cancel{q^2} + q'^2 - 2qq' - \cancel{q^2} - q'^2\cos^2\theta + 2qq'\cos\theta - q'^2\sin^2\theta = 0$$

$$\Rightarrow 2m_e c(q-q') + q'^2 - 2qq' - q'^2(\cos^2\theta + \sin^2\theta) + 2qq'\cos\theta = 0$$

$$\Rightarrow 2m_e c(q-q') + \cancel{q'^2} - 2qq' - \cancel{q'^2} + 2qq'\cos\theta = 0$$

$$\Rightarrow 2m_e c(q-q') - 2qq' + 2qq'\cos\theta = 0$$

$$\Rightarrow m_e c(q-q') = qq' - qq'\cos\theta$$

$$\Rightarrow m_e c(q-q') = qq'(1-\cos\theta)$$

$$\Rightarrow \frac{q-q'}{qq'} = \frac{1}{m_e c}(1-\cos\theta)$$

$$\Rightarrow \frac{1}{q'} - \frac{1}{q} = \frac{1}{m_e c}(1-\cos\theta) \longrightarrow (10)$$

We know that for a rest mass photon

$q = E/c = \frac{h\nu}{c}$ where $h$ is plank's Constant.

$\Rightarrow 1/q = c/h\nu$

Similarly, $\frac{1}{q'} = \frac{c}{h\nu'}$

Also, the relation between $\lambda$ and $\nu$ is given by

$$\lambda = \frac{c}{\nu}$$

so $\frac{1}{q} = \frac{\lambda}{h}$ ⟶ (11)

and $\frac{1}{q'} = \frac{\lambda'}{h}$

Subsituting (11) in (10), we get

$$\frac{\lambda'}{h} - \frac{\lambda}{h} = \frac{1}{m_e c}(1 - \cos\theta)$$

$$\Rightarrow \lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

$$\Rightarrow \lambda' = \lambda + \lambda_c (1 - \cos\theta) \longrightarrow (12)$$

Where $\lambda_c = \frac{h}{m_e c}$ is the compton wave

and it can be calculated as

$$\lambda_c = \frac{6.63 \times 10^{-34}}{(9.1 \times 10^{-31})(3 \times 10^{8})}$$

$$= 0.24 \times 10^{-11}$$

$$= 0.0024 \times 10^{-9}$$

$$= 0.0024 \text{ nm}$$

or $2.4 \times 10^{-12}$
$= 2.4$ pm
p ⟶ pico

Special Cases

(1) If $\theta = 0$; i.e. when light goes str

then $\lambda' = \lambda$

$\Rightarrow$ there is no change in wavelength.

② If $\theta = \frac{\pi}{2}$; i.e when the light (photon) is deflected at right angle, then

$$\lambda' = \lambda + \lambda_c$$

$\Rightarrow$ wavelength is increased exactly by the Compton wavelength.

③ If the light is sent back, i.e. $\theta = \pi$.

then $\lambda' = \lambda + 2\lambda_c$

$\Rightarrow$ wavelength is increased by two times the Compton wavelength.

Energy and Momentum of electron.

Will Shall find expression for momentum of the recoiling election in order to determine the energy of the election.

In particular $i = 1, 2$, eq ⑤ Can be written as

$i=1 \Rightarrow p_1' + q_1' = p_2' + q_2'$

$$\Rightarrow 0 + q = p\cos\varphi + q'\cos\theta$$

$$\Rightarrow p\cos\varphi = q - q'\cos\theta \quad \text{——— (13)}$$

For $i=2 \Rightarrow p_1^2 + q_1^2 = p_2^2 + q_2^2$

$$\Rightarrow 0 + 0 = -p\sin\varphi + q'\sin\theta$$

$$\Rightarrow p\sin\varphi = q'\sin\theta \quad \text{——— (14)}$$

Dividing (14) by (13), we get

$$\frac{p\sin\varphi}{p\cos\varphi} = \frac{q'\sin\theta}{q - q'\cos\theta}$$

$$\Rightarrow \tan\varphi = \frac{q'\sin\theta}{q\left(1 - \frac{q'}{q}\cos\theta\right)}$$

$$= \frac{\frac{1}{q}\sin\theta}{\frac{1}{q'} - \frac{1}{q}\cos\theta}$$

$$= \frac{\lambda/h \sin\theta}{\lambda'/h - \lambda/h \cos\theta} \quad \because \text{using } (11)$$

$$\Rightarrow \tan\varphi = \frac{\lambda\sin\theta}{\lambda' - \lambda\cos\theta}$$

$$= \frac{\lambda\sin\theta}{\lambda + \lambda_c(1-\cos\theta) - \lambda\cos\theta} \quad \text{using } (12)$$

$$= \frac{\lambda\sin\theta}{\lambda(1-\cos\theta) + \lambda_c(1-\cos\theta)}$$

$$= \frac{\lambda\sin\theta}{(\lambda+\lambda_c)(1-\cos\theta)}$$

$$= \frac{\cancel{\lambda}\sin\theta}{\cancel{\lambda}\left(1 + \frac{\lambda_c}{\lambda}\right)(1-\cos\theta)}$$

$$\Rightarrow \tan\varphi = \frac{\sin\theta}{\left(1+\frac{\lambda_c}{\lambda}\right)(1-\cos\theta)}$$

$$= \frac{\cancel{2}\sin\theta/2 \cos\theta/2}{\left(1+\frac{\lambda_c}{\lambda}\right)\cdot\cancel{2}\sin^{\cancel{2}}\theta/2}$$

$$= \frac{\cos\theta/2}{(1+\lambda_c/\lambda)\sin\theta/2} = \frac{\cot\theta/2}{1+\lambda_c/\lambda}$$

$$\Rightarrow \varphi = \tan^{-1}\left(\frac{\cot\theta/2}{1+\lambda_c/\lambda}\right) \quad (15)$$

Special Cases.

(1) In the case that $\lambda_c << \lambda$ we get the classical result.

$$\varphi \approx \tan^{-1}\left(\frac{\cot\theta/2}{1+0}\right) \qquad \because \lambda_c << \lambda \Rightarrow \frac{\lambda_c}{\lambda} \to 0$$

$$= \tan^{-1}(\cot\theta/2)$$

(2) when $\lambda_c >> \lambda$ we get

$$\varphi = \tan^{-1}\left(\frac{\cot\theta/2}{\frac{\lambda_c}{\lambda}\left(\frac{\lambda}{\lambda_c}+1\right)}\right)$$

$$\approx \tan^{-1}\left(\frac{\cot\theta/2}{\frac{\lambda_c}{\lambda}(0+1)}\right) \qquad \because \lambda_c >> \lambda \Rightarrow \frac{\lambda}{\lambda_c} \to 0$$

$$= \tan^{-1}\left(\frac{\cot\theta/2}{\lambda_c/\lambda}\right)$$

$$= \tan^{-1}\left(\frac{\lambda}{\lambda_c}\cot\theta/2\right)$$

(3) If $\theta = 0$ then

(15) $\Rightarrow \varphi = \tan^{-1}(\infty) = \pi/2$.

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If $\theta = \pi/2$, then

$$\textcircled{15} \Rightarrow \varphi = \tan^{-1}\left(\frac{\cot \pi/4}{1+\lambda_c/\lambda}\right) \qquad \because \cot\frac{\pi}{4} = 1.$$

$$= \tan^{-1}\left(\frac{1}{1+\lambda_c/\lambda}\right)$$

$$= \tan^{-1}\left(\frac{\lambda}{\lambda+\lambda_c}\right)$$

If $\theta = \pi$, then

$$\textcircled{15} \Rightarrow \varphi = \tan^{-1}\left(\frac{\cot \pi/2}{1+\lambda_c/\lambda}\right) = \tan^{-1}(0)$$

$$= 0$$

Now squaring Eq (13) and eq (14) and then adding, we get

$$p^2\cos^2\varphi + p^2\sin^2\varphi = (q - q'\cos\theta)^2 + (q'\sin\theta)^2$$

$$\Rightarrow p^2(\cos^2\varphi + \sin^2\varphi) = q^2 + q'^2\cos^2\theta - 2qq'\cos\theta + q'^2\sin^2\theta$$

$$\Rightarrow p^2 = q^2 + q'^2(\cos^2\theta + \sin^2\theta) - 2qq'\cos\theta$$

$$= q^2 + q'^2 - 2qq'\cos\theta$$

$$= q'^2\left(q^2/q'^2 + 1 - 2\frac{q}{q'}\cos\theta\right)$$

$$= q'^2\left[\left(\frac{q}{q'}\right)^2 + 1 - 2\left(\frac{q}{q'}\right)\cos\theta\right]$$

Taking square root, we get

$$p = q'\sqrt{1 + \left(\frac{q}{q'}\right)^2 - 2\left(\frac{q}{q'}\right)\cos\theta}$$

From eq (12), we have

$$\lambda' = \lambda + \lambda_c(1 - \cos\theta)$$

$$\Rightarrow \frac{h}{q'} = \lambda + \lambda_c(1-\cos\theta) \qquad \text{using (11)}$$

$$\Rightarrow q' = \frac{h}{\lambda + \lambda_c(1-\cos\theta)} \qquad (17)$$

Plugging eq (17) in eq (16), we get

$$p = \frac{h}{\lambda + \lambda_c(1-\cos\theta)}\sqrt{1 + \left(\frac{q}{q'}\right)^2 - 2\left(\frac{q}{q'}\right)\cos\theta} \longrightarrow (18)$$

Now we find $q/q'$ as follows.

$$\frac{q}{q'} = \frac{h/\lambda}{\dfrac{h}{\lambda+\lambda_c(1-\cos\theta)}} \qquad \text{using (11) and (17)}$$

$$= \frac{h(\lambda + \lambda_c(1-\cos\theta))}{h\lambda}$$

$$= 1 + \frac{\lambda_c}{\lambda}(1-\cos\theta)$$

Thus equation (18) becomes

$$p = \frac{h}{\lambda+\lambda_c(1-\cos\theta)}\sqrt{1 + \left(1 + \frac{\lambda_c}{\lambda}(1-\cos\theta)\right)^2 - 2\left(1+\frac{\lambda_c}{\lambda}(1-\cos\theta)\right)\cos\theta} \longrightarrow (19)$$

gives us momentum of the electron

Finally, the energy of electron is given by

$$E^2 = p^2c^2 + m_e^2 c^4$$

$$= p^2c^2\left(1 + \frac{m_e^2 c^2}{p^2}\right)$$

$$= p^2c^2\left(1 + \frac{h^2 m_e^2 c^2}{h^2 p^2}\right)$$

$$= p^2c^2\left(1 + \frac{h^2}{p^2} \cdot m_e^2 c^2/h^2\right)$$

$$= p^2c^2\left(1 + \left(\frac{h}{p}\right)^2 m_e^2 c^2/h^2\right)$$

$$= p^2c^2\left(1 + \lambda_e^2 m_e^2 c^2/h^2\right)$$

where $\lambda_e$ is the De Broglie wavele
of the electron.

$$\Rightarrow E = pc\sqrt{1 + \lambda_e^2 m_e^2 c^2/h^2}$$

$$= pc\sqrt{1 + \lambda_e^2/\lambda_c^2} \longrightarrow (2$$

where $\lambda_c = \frac{h}{m_e c}$ is Compton wav

Eq (20) including (19) gives us the ene
of electron.

(Ex) What is difference between laboratory 2012, 2014
and centre of mass frame. Also find the
expression for the energy and momentum
Components in the laboratory frame
the two particles.

## Particle Scattering.

### Laboratory frame.

The frame of reference in which the observe is performing the experiment is called laboratory frame.

### Centre of mass frame.

The frame in which the total momentum 4-vector has zero spatial components is called the centre of mass frame.

Let us consider two particles having 4-vector momenta $p_1^i$ and $p_2^i$ in the laboratory frame, and $p_1'^i$, $p_2'^i$ in the centre of mass frame.

Thus, the total momentum 4-vector in laboratory frame is given by

$$p^i = p_1^i + p_2^i$$
$$= (p^0, p^1, p^2, p^3)$$
$$= (E/c, \vec{p}) \longrightarrow (1)$$

The total momentum 4-vector in centre of mass frame is given by

$$p'^i = p_1'^i + p_2'^i$$
$$= (p'^0, p'^1, p'^2, p'^3)$$
$$= (E'/c, 0) \longrightarrow (2) \quad \because \vec{p} = \vec{0}$$

As we know that the relations for moment

$p$, and Energy $E$ are

$$\vec{p} = m\vec{v}$$

and $E = mc^2$

so $\dfrac{p}{E} = \dfrac{m\vec{V}}{mc^2}$ ——— (3)

$\Rightarrow \dfrac{p}{E} = \dfrac{\vec{V}}{c^2}$

In the centre of mass frame, the velocity is zero.

Thus $\vec{V}$ is the relative velocity between the centre of mass frame and the the laboratory frame.

The total momentum is the sum of 4-momentum vector of two particles

$$p'^i = p_1'^i + p_2'^i$$

$$= \left(p_1'^0 + p_2'^0,\ p_1'^1 + p_2'^1,\ p_1'^2 + p_2'^2,\ p_1'^3 + p_2'^3\right) \longrightarrow$$

Comparing (2) and (4), we get

$p_1'^1 + p_2'^1 = 0 \Rightarrow p_1'^1 = -p_2'^1$

$p_1'^2 + p_2'^2 = 0 \Rightarrow p_1'^2 = -p_2'^2$

and $p_1'^3 + p_2'^3 = 0 \Rightarrow p_1'^3 = -p_2'^3$

$\Rightarrow \vec{p}_1' = -\vec{p}_2' = \vec{p}'$ (say) ——— (5)

If the $x$-direction is taken to be along $\vec{V}$, then the L. transformations

relating the laboratory frame to the centre of mass frame are given by

$$P_{1x} = \gamma\left(P'_{1x} - \frac{V}{c} P_1'^{0}\right)$$

$$= \gamma\left(P'_{1x} - \frac{V}{c} \cdot E'_1/c\right)$$

$$\Rightarrow p_{1}x = \frac{P'_x - E'_1 V/c^2}{\sqrt{1 - V^2/c^2}} \qquad \because \text{using } (5)$$

$$P_{1y} = P'_{1y} = P'_y$$

$$P_{1z} = P'_{1z} = P'_z \qquad \text{using } (5)\ P'_1 = P'$$

and

$$P_{2x} = \gamma\left(P'_{2x} - \frac{V}{c} P_2'^{0}\right)$$

$$= \gamma\left(P'_{1x} - \frac{V}{c} \cdot \frac{E'_2}{c}\right)$$

$$= \frac{-P'_{2x} - E'_2 V/c^2}{\sqrt{1 - V^2/c^2}} \qquad \text{using } (5)$$

$$\text{or } P_{2x} = \frac{-P'_x - E'_2 V/c^2}{\sqrt{1 - V^2/c^2}}$$

$$P_{2y} = P'_{2y} = -P'_y \ ; \ P_{2z} = +P'_{2z} = -P'_z \qquad \text{using } (5)$$

The energy momentum of 1st particle is

$$E_1 = \gamma\left(E'_1 + P'_x V\right)$$

$$\Rightarrow E_1 = \frac{E_1' + p_x' V}{\sqrt{1 - V^2/c^2}}$$

and of second particle is

$$E_2 = \gamma(E_1' - p_x' V)$$

$$= \frac{E_1' - p_x' V}{\sqrt{1 - V^2/c^2}}$$

Example: 2017

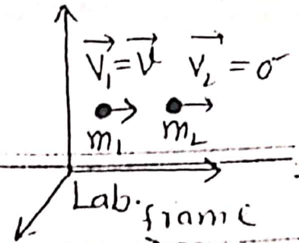
Describe particle scattering by considering two particles having four vector momenta in the laboratory frame, $p_1^\mu$ and $p_2^\mu$ with rest masses $m_1$ and $m_2$. Calculate the four momenta of these particles in the centre of mass frame.

Solution.

Let us consider a particle of rest mass $m_1$ and velocity $\vec{V_1} = \vec{V}$, colliding with a particle of mass $m_2$ at rest i.e. $\vec{V_2} = 0$ in laboratory frame.

In laboratory frame, we can write the expression of 4-momenta $p_1^\mu$ and $p_2^\mu$ and 3-momentum vectors $\vec{p_1}$ and $\vec{p_2}$.

Since the two particles are in motion in




laboratory frame.

Therefore, $\vec{P_1} = \gamma m_1 \vec{V_1}$ ; $\vec{P_2} = \gamma m_2 \vec{V_2}$

where $m_1$ and $m_2$ are proper masses of two particles.

$$\Rightarrow \vec{P_1} = \frac{m_1 \vec{V_1}}{\sqrt{1-\frac{\vec{V_1}\cdot\vec{V_1}}{c^2}}} \quad ; \quad \vec{P_2} = \frac{m_2 V_2}{\sqrt{1-\frac{\vec{V_2}\cdot\vec{V_2}}{c^2}}} = 0 \quad \because V_2 = 0$$

Thus the total momentum 4-vector is

$$p^{\mu} = (p^0, p^1, p^2, p^3)$$

$$= (p^0, \vec{P}) \qquad = (p_1^0 + p_2^0 , \vec{P_1} + \vec{P_2})$$

$$= \left(\frac{E_1}{c} + \frac{E_2}{c} , \frac{m_1 \vec{V_1}}{\sqrt{1-\frac{\vec{V_1}\cdot\vec{V_1}}{c^2}}} + 0\right)$$

$$= \left(\gamma m_1 c + m_2 c , \frac{m_1 \vec{V_1}}{\sqrt{1-\frac{\vec{V_1}\cdot\vec{V_1}}{c^2}}}\right) \qquad \because \begin{array}{l} m_1 \to \text{moving} \\ m_2 \to \text{at rest} \end{array}$$

$$= \left(m_2 c + \frac{m_1 c}{\sqrt{1-\frac{\vec{V_1}\cdot\vec{V_1}}{c^2}}} , \frac{m_1 \vec{V_1}}{\sqrt{1-\frac{\vec{V_1}\cdot\vec{V_1}}{c^2}}}\right) \qquad \text{— (1)}$$

Hence, we have

$$\vec{V} = \frac{m_1 \vec{V_1}}{m_1 + m_2 \sqrt{1-\frac{\vec{V_1}\cdot\vec{V_1}}{c^2}}} \qquad \text{— (2)}$$

gives the velocity of centre of mass frame

relative to the laboratory frame.
In this frame the Lorentz transformation of $p_2'$ is given by

$$p_2'^{\mu} = (p_2'^0, \vec{p}_2')$$

$$= \left( \frac{m_2 c}{\sqrt{1 - \frac{\vec{V}\cdot\vec{V}}{c^2}}}, \frac{-m_2 \vec{V}}{\sqrt{1 - \frac{\vec{V}\cdot\vec{V}}{c^2}}} \right) \quad \text{--- (3)}$$

Now, in centre of mass frame

$$\vec{p}_1' + \vec{p}_2' = 0$$

$$\Rightarrow \vec{p}_1' = -\vec{p}_2'.$$

$$\Rightarrow \vec{p}_1' = \frac{m_2 \vec{V}}{\sqrt{1 - \frac{\vec{V}\cdot\vec{V}}{c^2}}} \quad \text{--- (4)}$$

Also, the Lorentz transformation of energy gives

$$E_1' = \frac{E_1 - \vec{p}_1\cdot\vec{V}}{\sqrt{1 - \frac{\vec{V}\cdot\vec{V}}{c^2}}}$$

$$= \frac{\gamma m_1 c^2 - \gamma m_1 \vec{V}_1\cdot\vec{V}}{\sqrt{1 - \frac{\vec{V}\cdot\vec{V}}{c^2}}}$$

$$= \frac{\gamma m_1 c^2 \left[1 - \vec{V}_1\cdot\vec{V}/c^2\right]}{\sqrt{1 - \vec{V}\cdot\vec{V}/c^2}}$$

$$\Rightarrow E_1' = \frac{m_1 c^2 (1 - \vec{v}_1 \cdot \vec{V}/c^2)}{\sqrt{1 - \vec{v}_1 \cdot \vec{v}_1/c^2}\sqrt{1 - \vec{V} \cdot \vec{V}/c^2}} \quad \text{—— (5)}$$

Finally, we have

$$p_1'^{\mu} = (p_1'^{0}, \vec{p}_1') = \left(\frac{E_1'}{c}, \vec{p}_1'\right)$$

Using equation (4) and (5) in above equation, we get

$$p_1'^{\mu} = \left( \frac{m_1 c (1 - \vec{v}_1 \cdot \vec{V}/c^2)}{\sqrt{1 - \vec{v}_1 \cdot \vec{v}_1/c^2}\sqrt{1 - \frac{\vec{V} \cdot \vec{V}}{c^2}}}, \frac{m_2 \vec{V}}{\sqrt{1 - \vec{V} \cdot \vec{V}/c^2}} \right) \rightarrow (6)$$

Equations (3) and (6) including equation (2) give the 4-momenta in the centre of mass frame. A/2013, A/2016, A/2017, A/20

Determine a formula for the minimum K.E. required to produce a particle of mass m.

Particle production.

One of the consequences of the equivalence of mass and energy is the expectation that energy could be converted into mass not in the sense of increasing the moving mass of a single particle, but by creating more rest-mass in the form a new particle called particle production or particle creation.

For this purpose, let us consider a partic moving with certain velocity, we call this particle as projectile.
This particle strikes another particle. This particle moves off and the target particle moves as well.
We further assume that the collision of these two particles generates a new particle. Let us denote the rest-mass of the three particles by $m_p$ (for the projectile), $m_T$ (for the target particle) and $m_N$ (for the new particle produced).
For given masses of these particles, we want to calculate the minimum K.E. required to produce a new particle.
For this purpose, we find the energy and momentum of these particles in the laboratory frame and in the centre of mass frame.

① Energy and Momentum in laboratory frame.

Let $E$ be the energy of the projectile of mass $m_p$ in the laboratory frame. Then the 4-vector momenta of the projectile and the target particles

would be

$$p_P^{\mu} = (E/c, \vec{p}) \quad ; \quad p_T^{\mu} = (m_T c, 0) \quad \text{--- (1)}$$

where $p_P^{\mu}$ and $p_T^{\mu}$ denote the 4-vector momenta for projectile and target particle respectively.

Thus, the total 4-vector momentum before collision is

$$p^{\mu} = (p^0, p^1, p^2, p^3)$$
$$= (p^0, \vec{p})$$
$$= p_P^{\mu} + p_T^{\mu}$$
$$= (E/c + m_T c, \vec{p}) \quad \text{--- (2)}$$

Then squared magnitude in Minkowski space of $p^{\mu}$

$$g_{\mu\nu} p^{\mu} p^{\nu} = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$$
$$= (E/c + m_T c)^2 - |p|^2 \quad \text{--- (3)}$$
$$= E^2/c^2 + m_T^2 c^2 + 2 \frac{E}{c} m_T c - p^2$$

The energy mass relation is given by

$$E^2 = m^2 c^4 + p^2 c^2$$
$$\Rightarrow \frac{E^2}{c^2} = m^2 c^2 + p^2$$
$$\Rightarrow E^2/c^2 - p^2 = m^2 c^2 \quad \text{--- (4)}$$

which is the total momentum-energy before the collision.

In particular, for the projectile of mass

the relation ④ can be written as

$$\frac{E^2}{c^2} - p^2 = m_p^2 c^2 \quad \text{—— ⑤}$$

Subsituting ⑤ in ③, we get

$$g_{\mu\nu} p^\mu p^\nu = \left(\frac{E^2}{c^2} - p^2\right) + m_T^2 c^2 + 2 E m_T$$

$$= m_p^2 c^2 + m_T^2 c^2 + 2 E m_T$$

$$\Rightarrow g_{\mu\nu} p^\mu p^\nu = (m_p^2 + m_T^2) c^2 + 2 m_T E \text{ —}$$

⇒ total energy is invariant in laborato frame.

② Energy and Momentum in the centr of mass frame.

Let us denote the total energy of the system in centre of mass frame by $W$. Then the total 4-momentum in centre of mass frame is

$$p'^\mu = \left(\frac{W}{c}, \vec{p}'\right)$$

$$= \left(\frac{W}{c}, \vec{0}\right) \quad \because \vec{p}' = \vec{0} \text{ in CoM}$$

Again $g'_{\mu\nu} p'^\mu p'^\nu = (p'^0)^2 - (p'^1)^2 - (p'^2)^2 - (p'^3)^2$

$$= \frac{W^2}{c^2} - |\vec{p}'|$$

$$= \frac{W^2}{c^2} - \vec{0}$$

$$\Rightarrow g'_{\mu\nu} p'^{\mu} p'^{\nu} = W^2/c^2 \quad \text{——} \quad (7)$$

Lorentz invariance of squared magnitude of 4-vectors implies that the squared magnitude of 4-momentum vector in laboratory frame should be equal to 4-momentum vector in centre of mass frame.

$$\Rightarrow g'_{\mu\nu} p'^{\mu} p'^{\nu} = g_{\mu\nu} p^{\mu} p^{\nu} \quad \because g_{\mu\nu} = g'_{\mu\nu}$$

in Mink. space

Subsituting (6) and (7) in above relation, we get

$$\frac{W^2}{c^2} = (m_p^2 + m_T^2)c^2 + 2m_T E$$

$$\Rightarrow \frac{W^2}{c^2} - (m_p^2 + m_T^2)c^2 = 2m_T E$$

$$\Rightarrow E = \frac{W^2/c^2 - (m_p^2 + m_T^2)c^2}{2m_T} \quad \text{——} \quad (8)$$

which is total energy of the projectile in laboratory frame.

To find the mass $m_N$, for new particle, we can write the final mass as

$$m_f = m_p + m_T + m_N \quad \text{——} \quad (9)$$

where $m_f$ denotes final mass.

The K.E. of the projectile, T, is given by

$$T = E - m_p c^2 \quad (10)$$

Subsituting (8) in (10), we get

$$T = \frac{W^2/c^2 - (m_p^2 + m_T^2) c^2}{2m_T} - m_p c^2$$

$$= \frac{W^2/c^2 - (m_p^2 + m_T^2) c^2 - 2 m_T m_p c^2}{2m_T}$$

$$= \frac{W^2/c^2 - (m_p^2 + m_T^2 + 2 m_p m_T) c^2}{2m_T}$$

$$= \frac{W^2/c^2 - (m_p + m_T)^2 c^2}{2m_T} \quad (11)$$

Let us introduce initial mass by $m$

as $m_i = m_p + m_T$ (12)

Then, equation (11) becomes

$$T = \frac{W^2/c^2 - (m_i)^2 c^2}{2m_T}$$

$$= \frac{W^2 - m_i^2 c^4}{2m_T c^2} \longrightarrow (13)$$

In centre of mass frame $E^2 = m^2c^4 +$

Can be written as

$$W^2 = m_f^2 c^4 + (0)^2 c^2$$

$$\Rightarrow W^2 = m_f^2 c^4 \quad (14)$$

The threshold energy, which is the least K.E. denoted by $T_0$ can be determined by putting (14) in (13), we get

$$T_0 = \frac{m_f^2 c^4 - m_i^2 c^4}{2 m_T c^2}$$

$$= \frac{(m_f^2 - m_i^2) c^4}{2 m_T c^2}$$

$$\Rightarrow T_0 = \frac{(m_f^2 - m_i^2) c^2}{2 m_T}$$

which is required result.

Particle Decay.

It is the reverse process of particle production in which an elementry particle broken down into two or more numbes of elementry particles. If the created particles are not stable, then decay process can continue.

Let us assume that a particle of total mass $M$, travelling along with a momentum 4- vector

$$p^\mu = (p^0, p^1, p^2, p^3)$$

$$= (p^0, \vec{p})$$

$$= (E/c, \vec{p}) \quad (1)$$

Suddenly breaks into two particles of rest masses $m_1$ and $m_2$ and momentum 4- vectors

(10)

$$p_1^{\mu} = (p_1^0, p_1^1, p_1^2, p_1^3)$$
$$= (p_1^0, \vec{p}_1)$$
$$= (E_1/c, \vec{p}_1) \quad \text{—— (2)}$$

and

$$p_2^{\mu} = (p_2^0, p_2^1, p_2^2, p_2^3)$$
$$= (p_2^0, \vec{p}_2)$$
$$= (E_2/c, \vec{p}_2) \quad \text{—— (3)}$$

In the rest frame of initial particle the momentum is zero. Thus, the cent of mass frame of the final particles is the rest frame of the initial partic It has the velocity relative to the laboratory frame which is given by

$$\vec{P} = (\text{mass})\,\vec{V}$$
$$= (E/c^2)\,\vec{V} \qquad \because E = (\text{mass}$$
$$\Rightarrow \vec{P}c^2 = E\vec{V}$$
$$\Rightarrow \vec{V} = \frac{\vec{P}c^2}{E} \quad \text{—— (4)}$$

In the centre of mass frame, the initia momentum 4-vector is

$$P'^{\mu} = (P'^0, P'^1, P'^2, P'^3)$$
$$= (P'^0, \vec{P}')$$
$$= (Mc, \vec{0}) \qquad \because \vec{P} = \vec{0} \text{ in CO fra}$$

$$\Rightarrow P'^{\mu} = (Mc, \vec{0}) \quad \text{——— (5)}$$

The individual 4-vector momentum are

$$P_1'^{\mu} = (E_1'/c, \vec{P}_1') \; ; \; P_2'^{\mu} = (E_2'/c, \vec{P}_2') \quad \text{——— (6)}$$

But $P'^{\mu} = P_1'^{\mu} + P_2'^{\mu}$

$$\Rightarrow P'^{\mu} = (P_1'^{0} + P_2'^{0}, P_1'^{1} + P_2'^{1}, P_1'^{2} + P_2'^{2}, P_1'^{3} + P_2'^{3}) \longrightarrow (7)$$

From relation (5) and (7), we can write

$$P_1'^{0} + P_2'^{0} = Mc$$

$$\Rightarrow E_1'/c + E_2'/c = Mc$$

$$\Rightarrow E_1' + E_2' = Mc^2 \quad \text{——— (8)}$$

and $\vec{P}_1' + \vec{P}_2' = \vec{0}$

$$\Rightarrow \vec{P}_1' = -\vec{P}_2' = \vec{P} \quad (\text{say}) \quad \text{——— (9)}$$

If the rest masses of decay products are $m_1, m_2$, then

$$E_1'^2 = m_1^2 c^4 + P_1'^2 c^2 \longrightarrow (10)$$

and $E_2'^2 = m_2^2 c^4 + P_2'^2 c^2$

Subsituting (9) in (10), we get

$$E_1'^2 = m_1^2 c^4 + \vec{P}^2 c^2 \qquad \because \vec{P}_1' = \vec{P}$$

and $E_2'^2 = m_2^2 c^4 + \vec{P}^2 c^2 \qquad \because \vec{P}_2' = \vec{P}$

$$\Rightarrow E_1'^2 = (m_1^2 c^2 + \vec{P}^2) c^2 \text{ and } E_2'^2 = (m_2^2 c^2 + \vec{P}^2) c$$

$$\Rightarrow E_1' = \sqrt{\vec{p}'^2 + m_1'^2 c^2}\; c \qquad (11)$$

and $$E_2' = \sqrt{\vec{p}'^2 + m_2'^2 c^2}\; c$$

Since there are three equations for the six parameters $M$, $p$, $E_1'$, $E_2'$, $m_1'$ and $m_2'$, therefore, we need to know three of them to determine the other three. Once these are worked out, the relevant 4-vectors can be transformed by the Lorentz transformation given by $\vec{V}$ as mentioned in equation (4).